Advanced Curve Sketching

We have already seen how we can use calculus to graph relatively simple functions that have domains which include all values of $x$. These functions are all continuous.

Not every function that we will deal with will have continuity, some will have discontinuities.

Examples:

$$f(x) = \frac{1}{x^2}$$

Where is the discontinuity?

$x = 0$

$x$ can never equal 0 because that would make 0 in the denominator. This is called a vertical or $x$-asymptote. The function gets infinitely close to, but never touches the line $x = 0$.

What happens to the function at the point $x = 0$?

It goes from increasing to decreasing.

So, the derivative of $f(x)$ goes from being positive to negative at the point $x = 0$.

What is the derivative of the function at $x = 0$?

$$f(x) = \frac{x^2 - 3}{x^3}$$

$$f'(x) = -\frac{2x^2 - 9}{x^4}$$

At $x = 0$, it does not exist.
So, before we said that increasing/decreasing can only change around a critical point.

We need to revise this statement and say that the function can change from inc/dec at critical points AND vertical asymptotes.

\[ f(x) = x^2 \]
\[ f'(x) = 2x \]

Critical Points: None

\[
\begin{array}{c|c|c}
\text{c} & \text{f'(c)} & \text{Min/Max} \\
(0) & 0 & \text{None} \\
(1) & 2 & \\
\end{array}
\]

Increasing to decreasing

**Visual Example 2**

\[ f(x) = \frac{x^2 + 1}{x - 1} \]

Where is the discontinuity?

\[ x = 1 \]

We can see that the discontinuity is there by looking at the fraction.

What happens at the discontinuity? It does not change from decreasing to increasing. The concavity changes from concave down to concave up.
This means that what happens to the second derivative around the point $x=1$?

It goes from negative to positive about the point $x=1$.

$f(x) = \frac{x^2 - 1}{x - 1}$

$f'(x) = \frac{(x-1)(2x) - (x^2 - 1)}{(x-1)^2}$

$= \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2}$

$= \frac{x^2 - 2x - 1}{(x-1)^2}$

$f''(x) = \frac{(x-1)^2(2x-2) - (x^2 - 2x - 1)2(x-1)}{(x-1)^4}$

$= \frac{(x-1)[(x-1)(2x-2) - 2(x^2 - 2x - 1)]}{(x-1)^4}$

$= \frac{(x-1)[2x^2 - 2x - 2x + 2 - 2x^2 + 4x + 2]}{(x-1)^4}$

$= \frac{(x-1)(4)}{(x-1)^4} \frac{4}{(x-1)^3}$

$F''(x) \begin{cases} 
\infty & (0) \\
1 & (2) \\
\ast & C.D. \\
\ast & C.U.
\end{cases}$

So, before we said that concavity can only change at inflection points, but we need to add that concavity can also change around asymptotes.
So, the point is when it comes to curve sketching, we must add a step and revise some.

1. Y-intercepts (and x-intercepts if practical)
2. Look for x-asymptotes (vertical asymptotes)
3. Critical Points: \( f'(x) = 0 \)
4. Inc/Dec. - First derivative test with critical points and asymptotes.
5. Inflection Points: \( f''(x) = 0 \)
6. Second derivative test - inflection points and asymptotes.
7. Draw

Example: \( f(x) = \frac{x^4 + 4}{x + 3} \)

Let's graph this function:

1. Intercepts
2. Vertical Asymptote: \( x = -3 \)
3. \( f'(x) = \frac{(x+3)-(x+4)}{(x+3)^2} = \frac{-1}{(x+3)^2} \)
4. Critical Points (None)
5. \( f''(x) = \frac{2(x+3)^2 - 2(x+3)}{(x+3)^4} \)
6. Inflection Points (None)
7. Draw

But how do we know what this looks like?

How do we know this doesn't happen? (No other x asymptote?)
We need to know what happens to the function at the extremes.

\[ f(x) = \frac{x + 4}{x + 3} \]

There are two ways to do this, a practical way and a theoretical way.

**Practical:** Choose a really negative number and put it in the function and choose a really positive number and put it in the function.

\[
\begin{align*}
 f(-1000) &= \frac{-996}{-997} \approx 1 \\
 f(-10,000) &= \frac{-9996}{-9997} \approx 1 \\
 f(1000) &= \frac{1004}{1003} \approx 1 \\
 f(10,000) &= \frac{10004}{10003} \approx 1
\end{align*}
\]

Never gets to \( y = 1 \)

**Theoretical:**

\[
\begin{align*}
 \lim_{x \to -\infty} \frac{x + 4}{x + 3} &= \frac{-\infty}{-\infty} = 1 \\
 \lim_{x \to \infty} \frac{x + 4}{x + 3} &= \frac{\infty}{\infty} = 1
\end{align*}
\]

\( \text{(Infinity plus 4 is nothing)} \)

Two y-asymptotes (left and right)

Now we can draw this function.
So now we must revise again our rules for curve sketching.

1. Intercepts (y and x if practical)
2. Vertical x-intercepts
3. Horizontal y-intercepts
4. Critical Points, \( f'(x) = 0 \)
5. FDT (Inc/Dec) Include CP/Asympt.
6. Inflection Points, \( f''(x) = 0 \)
7. SIT (C.U./C.D) Include Inflection Pts/Asympt.
8. Graph
9. Now, do the happy dance.

A word on horizontal Asymptotes

\[
\lim_{x \to \infty} \frac{3x^4 + 4x - 2}{2x^2}
\]

In this case, we only need to put infinity into the highest degree variable in the polynomial.

In the numerator, the \( +4x - 2 \) is just like the \( +4 \) in \( x + 4 \). It becomes irrelevantly small as \( x \) gets bigger.

\[
\lim_{x \to \infty} \frac{3x^2}{2x^2} = \frac{3}{2}
\]

We can check this by putting a large number into the equation.

\[
f(1000) = \frac{3(1000)^2 + 4(1000) - 2}{2(1000)^2} \approx \frac{3}{2}
\]
\[ f(x) = \frac{1}{x^2 + 9} \]

**Vertical Asymptotes**

\[ x = 3, -3 \]

**Horizontal Asymptote**

\[ \lim_{x \to \pm \infty} f(x) = \frac{1}{x^2} \]

\[ f'(x) = \frac{(x^2 - 9)(0) - (1)(2x)}{(x^2 - 9)^2} \]

\[ = \frac{-2x}{(x^2 - 9)^2} \]

\[ x = 0 \quad (0, \frac{1}{9}) \quad \text{Crit Pt.} \]

\[ f''(x) = \frac{x^3}{(x^2 - 9)^2} - \frac{(3)}{(2)}(2x)(2)(x^2 - 9) \]

\[ = \frac{2x^3 + 18x}{x^2 - 9} \]

\[ f''(x) \cdot f'(x) = \frac{2x^3 + 18x}{x^2 - 9} \cdot \frac{-2x}{(x^2 - 9)^2} \]

\[ \text{No Inflection Points.} \]

\[ f'(x) > 0 \quad \text{Max} \quad f'(x) < 0 \quad \text{Min} \]

\[ f''(x) \cdot f'(x) > 0 \quad \text{CU} \quad f''(x) \cdot f'(x) < 0 \quad \text{CD} \]

\[ (4) \quad (0) \quad (3) \quad (4) \]

**Now Draw**
Shifting Focus ~ Using Calculus Practically ~ Optimization

This is the last part of the uses for calculus:
✓ 1) Approximation
✓ 2) Marginal Analysis
✓ 3) Curve Sketching
→ 4) Optimization

\[ f(x) = x^4 - 3x^2 + 1 \]

We have already gathered that min and max occur at critical points. \( f'(x) = 0 \)

If I just gave you the functions, and you didn’t draw it, how would we know if a crit pt was a max or a min?

Check Second order conditions:
A max always occurs during concave down
A min always occurs at concave up

So, at the crit point \( x = 2 \), what must be true about the second derivative at that point?
\[ f''(x) = 12x + 0 \]
\[ f''(2) = \text{negative} \rightarrow \text{MUST BE CONCAVE DOWN (MAX)} \]
\[ f''(0) = \text{positive} \rightarrow \text{MUST BE CONCAVE UP (MIN)} \]
\[ f'(x) = 0 \text{ and } f''(x) = + \quad \text{MIN} \]
\[ f'(x) = 0 \text{ and } f''(x) = - \quad \text{MAX} \]
\[ f'(x) = 0 \text{ and } f''(x) = 0 \quad \text{NEITHER} \]

**Example:**
\[ f(x) = x^2 + 4x + 5 \]
\[ f'(x) = 2x + 4 \]
\[ x = -2 \]
\[ f''(x) = 2 \quad \text{positive (curve up)} \]
\[ (-2, 1) \]
\[ \text{MIN} \]

Is this a global min or just a local min.

\[ \lim_{x \to -\infty} f(x) = +\infty \]
\[ \lim_{x \to +\infty} f(x) = +\infty \]

**GLOBAL MIN**

Let's look at our previous example:
\[ f(x) = x^3 - 3x^2 + 1 \]

We know that there are local mins and maxes.

Let's say we are interested in the function \( f(x) = x^3 - 3x^2 + 1 \) in the interval \(-5 \leq x \leq 5\).

The max/min could occur at points that are NOT detected by critical points.

How can we figure out what the min/max is?
\[ f(x) = x^3 + 2x^2 + 1 \]
\[ f(0) = (0)^3 + 2(0)^2 + 1 = 1 \]
\[ f(5) = (5)^3 + 2(5) + 1 = 126 \]
\[ -125 + 51 = -74 \]
\[ (-5, -74) \]
\[ f(5) = (5)^3 + 51 = (5, 174) \]

The min and maxes are at the boundary.

So, in bounded optimization, we must find the critical points, check the order condition, and boundary points.

In unbounded optimization, we find critical points, check the order condition, and the boundary points are infinite.

\[ f(t) = 3t^3 - 5t^2 - 2 \geq 0 \]
\[ f'(t) = 9t^2 - 10t \]
\[ f'(0) = 10(0)(1-1) = 0 \]
\[ f''(t) = 18t - 10 \]
\[ f''(0) = -10 \]
\[ f''(-1) = -10 \]

Check boundary points:
\[ f(-2) = -56 \]
\[ f(0) = 0 \]

Example:

\[ f(t) = 3t^3 - 5t^2 \]
\[ f''(t) = 18t - 10 \]
\[ f''(t) = 0 \]
\[ t = \frac{5}{9} \]
\[ f''(0) = -10 \]

Not interested in bounds.

\[
\begin{cases}
(-1, 2) & \text{max} \\
(0, 0) & \text{min} \\
(1, 2) & \text{min}
\end{cases}
\]

\[
\begin{cases}
(-1, 2) & \text{max} \\
(0, 0) & \text{min}
\end{cases}
\]

Check boundary points:
\[ f(-2) = -56 \]
\[ f(0) = 0 \]