THE NOETHER–LEFSCHETZ THEOREM IN ARBITRARY CHARACTERISTIC

LENA JI

ABSTRACT. We show that if $X \subset \mathbb{P}_k^N$ is a normal variety of dimension $n \geq 3$ and $H \subset \mathbb{P}_k^N$ a very general hypersurface of degree $d = 4$ or ≥ 6, then the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(X \cap H)$ is an isomorphism up to torsion. If $n \geq 4$, the result holds for $d \geq 2$. The proof uses the relative Jacobian of a curve fibration, together with a specialization argument, and the result holds over fields of arbitrary characteristic.

CONTENTS

1. INTRODUCTION

For a normal variety X, let $\mathrm{Cl}(X)$ be its Weil divisor class group. The main result of this article is a Noether–Lefschetz result for Cl:

Theorem 1.1 (Proposition 3.1, Proposition 4.1)**.** *Let $k \neq \overline{\mathbb{F}}_p$ be an algebraically closed field of arbitrary characteristic $p \geq 0$, X a normal projective variety of dimension $n \geq 3$ over k, $\mathcal{L}_0$ a very ample line bundle on X, and d an integer. For $Y \in |\mathcal{L}_0^d|$ the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

(1) is injective for $d \geq 2$ and general Y,
(2) is surjective up to torsion for $d = 4$ or ≥ 6 and very general Y if $n = 3$, and
(3) is surjective up to p-power torsion for $d \geq 2$ and very general Y if $n \geq 4$.

Here very general means contained in a nonempty subset that is the complement of a countable union of proper closed subvarieties of $|\mathcal{L}_0^d|$. In (3), the restriction map is surjective under the additional assumption that either char $k = 0$ *or* k *has infinite transcendence degree over* $\mathbb{F}_p$. *(1) also holds over* $\overline{\mathbb{F}}_p$.

The classical Noether–Lefschetz theorem, stated by M. Noether [Noe82] and proven by Lefschetz [Lef21] over the complex numbers, says that if $S_d \subset \mathbb{P}^3_{\mathbb{C}}$ is a very general surface of degree $d \geq 4$, then the restriction map $\operatorname{Pic}(\mathbb{P}^3_{\mathbb{C}}) \to \operatorname{Pic}(S_d)$ is an isomorphism. In the classical statement, very general means away from a countable union of proper closed subvarieties of the parameter space $|\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(d)|$, referred to as the Noether–Lefschetz locus. Moreover, Lefschetz also showed for $N \geq 3$ that $\operatorname{Pic}(\mathbb{P}^N_{\mathbb{C}}) \to \operatorname{Pic}(S)$ is an isomorphism for a very general complete intersection surface $S \subset \mathbb{P}^N_{\mathbb{C}}$ unless S is the intersection of quadric threefolds in $\mathbb{P}^4_{\mathbb{C}}$ or a degree 2 or 3 surface in $\mathbb{P}^3$. There have been many generalizations of the classical theorem in numerous directions over the complex numbers [CGGH83, Gre84, Ein85, Voi88, Voi89], and for arbitrary smooth ambient threefolds X over $\mathbb{C}$ the Noether–Lefschetz theorem is known to hold for very general $Y \in |\mathcal{L}|$ with certain homological assumptions or if the line bundle $\mathcal{L}$ is sufficiently positive relative to K_X [Moi67, Voi03, Jos95, BN20].

In higher dimensions, the results are stronger: the Grothendieck–Lefschetz theorem in characteristic 0 says that if X is smooth of dimension $n \geq 4$ then for any ample effective divisor Y the restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ is an isomorphism, and injectivity still holds if $n = 3$ [Gro05, Har70].

Over an algebraically closed field k of arbitrary characteristic, Deligne showed using ℓ-adic cohomology that the classical Noether–Lefschetz theorem for surfaces in $\mathbb{P}^3_k$ and Lefschetz's result for complete intersection surfaces in $\mathbb{P}^N_k$ hold [DK73]. If k is uncountable then the complement of the Noether–Lefschetz locus automatically contains a point, and this still holds even if $k \neq \overline{\mathbb{F}}_p$ is countable [Ter85], and for $\overline{\mathbb{F}}_p$ when $p \equiv 1 \pmod{(d-1)^3+1}$ and $(d, 6) = 1$ [Shi83, Theorem 5.1]. If the degree d of the surface is even, however, the Tate conjecture implies that every smooth S_d has even Picard number, so the Noether–Lefschetz locus covers $|\mathcal{O}_{\mathbb{P}^3_{\overline{\mathbb{F}}_p}}(d)|(\overline{\mathbb{F}}_p)$. In particular this happens for $d = 4$ and $p \geq 5$ [Cha13]. In positive characteristic, Grothendieck–Lefschetz holds for complete intersections in $\mathbb{P}^N_k$, and in general for smooth varieties up to finite p-power torsion by replacing Kodaira vanishing with asymptotic Serre vanishing in the proof.

Another closely related problem is the specialization of Néron–Severi groups. If $X \to B$ is a smooth proper morphism of varieties over an algebraically closed field k, then the specialization map $\operatorname{NS}(X_{\overline{\eta}}) \to \operatorname{NS}(X_b)$ from the geometric generic fiber to a closed fiber is always injective up to torsion, and the locus where the Picard rank jumps is a countable union of proper subvarieties of B. If $k \neq \overline{\mathbb{F}}_p$ then the complement of this jumping locus contains a point [And96, MP12, Amb18, Chr18].

There are also results for singular ambient varieties in characteristic 0. For normal varieties, Noether–Lefschetz for Picard groups no longer holds, but Ravindra and Srinivas proved a variant for class groups: if $\mathcal{L}$ is a basepoint free and ample line bundle on a normal threefold X and if $K_X \otimes \mathcal{L}$ is globally generated then the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism for very general $Y \in |\mathcal{L}|$ [RS09]. Again passing to a resolution, they use techniques first introduced by Srinivas and Kumar and show a formal Noether–Lefschetz theorem on the universal family of divisors. The Noether–Lefschetz problem has also been studied by Bruzzo and Grassi for complete simplicial toric threefolds [BG12, BG18] and by Bruzzo, Grassi, and Lopez for $\mathbb{Q}$-factorial normal threefolds with rational singularities [BGL20]. In higher dimensions, Ravindra and Srinivas proved Grothendieck–Lefschetz for class groups, showing that for any basepoint free and ample line bundle $\mathcal{L}$ on a normal variety X of dimension ≥ 4, $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is an isomorphism for general $Y \in |\mathcal{L}|$ [RS06], by adapting Grothendieck's strategy to a resolution of singularities of X. Brevik and Nollet generalized this to hypersurface sections of X containing a fixed base locus [BN16].

To prove Theorem 1.1, we will study the divisors on X using a family of abelian varieties—namely the relative Jacobian of a general curve fibration. Néron used this approach in his thesis [Nér52], where he related divisors on X to the Jacobian of the generic fiber in his proof of the Theorem of the Base. This idea of fibering by curves dates back to Picard, and it has also been used to construct the Picard variety [Mat52, Igu52, Cho52].

1.1. **Outline.** The proof of Theorem 1.1 uses two main ingredients: the relative Jacobian and a degeneration argument. We being by showing the degeneration argument for quartic surfaces in $\mathbb{P}^3$ in §1.2. Section 2 contains preliminaries, including a Bertini result for sufficiently ample line bundles in §2.4 which will inform the situation in which we consider the relative Jacobian.

Section 3 contains the proof of injectivity. We study sections of the relative Jacobian in §3.1. In §3.2 we show that the subgroup of Cl^0 of algebraically trivial divisors is the same for a general $Y \in |\mathcal{L}|$. For injectivity of Cl, we consider specialization of sections of this family and show that the restriction map to "many" complete intersection curves is injective in §3.3.

We show surjectivity in Section 4. In §4.1 we cut to surfaces instead of curves and apply a result of Graber–Starr [GS13] on extending sections of families of abelian varieties to get surjectivity to a very general reducible complete intersection surface $S_1 + S_2$. In Proposition 4.7 we specialize from a very general T in $|\mathcal{L}^2|$ to $S_1 + S_2$ and show that the restriction map to T is surjective modulo torsion. For odd multiples of $\mathcal{L}$ we use a different type of degeneration to a reducible member in Proposition 4.9. These arguments work over a ground field that has infinite transcendence degree over the prime subfield $\mathbb{Q}$ or $\mathbb{F}_p$. Finally, in §4.2 we show the existence of a divisor over any ground field $k \neq \overline{\mathbb{F}}_p$ for which surjectivity holds.

1.2. **Quartic surfaces in $\mathbb{P}^3$.** In the case when $X = \mathbb{P}^3$ and $\mathcal{L}_0^d = \mathcal{O}_{\mathbb{P}^3}(4)$, the argument of Section 4 becomes very explicit (and does not require the assumption that the ground field has infinite transcendence degree). So we begin by showing the argument for quartic surfaces in $\mathbb{P}^3$.

Let $K \not\subset \overline{\mathbb{F}}_p$ be a field, $C \subset \mathbb{P}^3_K$ a degree 4 elliptic curve of rank at least 18, and $q_1, q_2, p_1, \dots, p_{15} \in C(K)$ points that are independent in $\mathrm{Pic}(C)/\langle H\rangle$, where H is the restriction of the hyperplane class on $\mathbb{P}^3$. For each $i = 1, 2$, the degree 2 line bundle $2q_i$ on C uniquely determines a smooth quadric surface $Q_i = (f_i = 0)$ such that the image of the restriction $\mathrm{Pic}(Q_i) \to \mathrm{Pic}(C)$ is generated by $2q_i$ and $H - 2q_i$. The points $p_1, \dots, p_{15}$ determine a quartic surface $T = (g = 0)$ and a unique sixteenth point $p_{16} \in C(K)$ such that $T \cap C = p_1 + \cdots + p_{16}$.
Claim: For any $\lambda \notin \overline{K}$, the quartic surface defined by $f_1 f_2 + \lambda g = 0$ has Picard rank 1.

Proof. Let $|\Lambda| \cong \mathbb{P}^1$ be the pencil in $|\mathcal{O}_{\mathbb{P}^3}(4)|$ spanned by $Q_1 + Q_2$ and T, and let $\tilde{X} = \{s_0 f_1 f_2 + s_1 g = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1{}_{[s_0 :\, s_1]}$ be the total space of the pencil.

Now consider the Chow variety $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1)$ of relative Weil divisors [Kol17, 3.21.3]. The components of $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1)$ are defined over $\overline{K}$, so for any $\lambda \notin \overline{K}$ the point $[1 \colon \lambda] \in \mathbb{P}^1$ will not be in the image of any component of $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1)$ not dominating the base. Let $T_\lambda = (f_1 f_2 + \lambda g = 0)$ be the corresponding quartic surface, and let D_λ be a curve on T_λ. After a base change by a smooth curve $\Gamma \to \mathbb{P}^1$ we can find a divisor $\mathcal{D}$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ whose fiber over $\tilde{\lambda}$ is D_λ, where $\tilde{\lambda} \in \Gamma(\overline{K(\lambda)})$ maps to $[1 \colon \lambda] \in \mathbb{P}^1(\overline{K(\lambda)})$; see the proof of Proposition 4.7 for details.

Let $\tilde{0} \in \Gamma$ be a point mapping to $[1 : 0] \in \mathbb{P}^1(K)$, and denote the fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{0}} = \tilde{Q}_1 + \tilde{Q}_2$, so that $\tilde{Q}_i \xrightarrow{\cong} Q_i$ and $\tilde{C} := \tilde{Q}_1 \cap \tilde{Q}_2 \xrightarrow{\cong} C$. By a local computation (Lemma 4.8), the class group of the local ring of $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ at a point $\tilde{x}$ in $\tilde{C}$ is either

- $\mathbb{Z}/r\mathbb{Z}$ if $\tilde{x}$ maps to a point of $C \setminus \{p_1, \dots, p_{16}\}$ and r is the ramification index of $\Gamma \to \mathbb{P}^1$ at $\tilde{0}$, or
- $\mathbb{Z}$, generated by $(\tilde{Q}_1)_{\tilde{x}}$ (or equivalently $(-\tilde{Q}_2)_{\tilde{x}}$), if $\tilde{x}$ maps to some p_j.

Let $\tilde{p}_j \in \tilde{C}$ be the point mapping to p_j for $1 \le j \le 16$. Then $\mathcal{D}$ uniquely determines a Weil divisor class D_i° on $\tilde{Q}_i$, and the restriction $D_1^\circ|_{\tilde{Q}_1 \cap \tilde{Q}_2} - D_2^\circ|_{\tilde{Q}_1 \cap \tilde{Q}_2}$ is supported on $\{\tilde{p}_1, \dots, \tilde{p}_{16}\}$. Write $D_1^\circ|_{\tilde{Q}_1 \cap \tilde{Q}_2} - D_2^\circ|_{\tilde{Q}_1 \cap \tilde{Q}_2} = \sum_{j=1}^{16} a_j \tilde{p}_j$. Then the divisor $\sum_{j=1}^{16} a_j p_j$ on C is in the subgroup $\langle H, 2q_1, 2q_2\rangle$ of $\mathrm{Pic}(C)$ generated by the images of $\mathrm{Pic}(Q_i) \to \mathrm{Pic}(C)$, so for some $b_1, b_2 \in \mathbb{Z}$ we have

$$2b_1 q_1 + 2b_2 q_1 + \sum_{j=1}^{16} a_j p_j \sim 2b_1 q_1 + 2b_2 q_1 + 4a_{16} H + \sum_{j=1}^{15} (a_j - a_{16}) p_j \in \langle H\rangle.$$

Since $q_1, q_2, p_1, \ldots, p_{15}$ are independent in $\mathrm{Pic}(C)/\langle H\rangle$, this implies $-2b_1 = -2b_2 = a_j - a_{16} = 0$ for $1 \le j \le 15$, so

$$D_1^\circ|_{\tilde{Q}_1\cap\tilde{Q}_2} - D_2^\circ|_{\tilde{Q}_1\cap\tilde{Q}_2} \sim a_{16}\sum_{j=1}^{16}\tilde{p}_j.$$

Then the divisor $\mathcal{D} - a_{16}\tilde{Q}_1$ on $\tilde{X}\times_{\mathbb{P}^1}\Gamma$ is $\mathbb{Q}$-Cartier along $\tilde{C}$. Let D_i be the divisor on $\tilde{Q}_i$ defined by $\mathcal{D} - a_{16}\tilde{Q}_1$. By abuse of notation we also denote by D_i the corresponding divisor under $\tilde{Q}_i \xrightarrow{\cong} Q_i$. Since $\mathrm{Pic}(Q_1)\times_{\mathrm{Pic}(C)} \mathrm{Pic}(Q_2) \cong \mathbb{Z}$ is generated by the restriction of the hyperplane class on $\mathbb{P}^3$, there is some $d \in \mathbb{Z}$ such that $D_1 \in |\mathcal{O}_{\mathbb{P}^3}(d)|_{Q_1}|$ and $D_2 \in |\mathcal{O}_{\mathbb{P}^3}(d)|_{Q_2}|$.

Now on $\tilde{X}\times_{\mathbb{P}^1}\Gamma$, the difference between the pullback of $\mathcal{O}_{\mathbb{P}^3}(d)$ and $\mathcal{D} - a_{16}\tilde{Q}_1$ is a divisor that is Cartier on the generic fiber, $\mathbb{Q}$-Cartier along the fiber $\tilde{Q}_1 + \tilde{Q}_2$ over $\tilde{0} \in \Gamma$, and trivial when restricted to $\tilde{Q}_1 + \tilde{Q}_2$. So by injectivity of specialization of Néron–Severi groups, the fiber over $\tilde{\lambda}$ is in $\mathrm{Cl}^0(T_\lambda) = 0$. That is, $D_\lambda \in |\mathcal{O}_{\mathbb{P}^3}(d)|_{T_\lambda}|$.

□

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2. Preliminaries

2.1. **Class groups of normal varieties.** Let X be a proper, normal variety over an algebraically closed field k. When we say divisor without specifying further, we refer to Weil divisors. $\sim$ will denote linear equivalence of divisors, and if D is a divisor on X, then by abuse of notation we will also write $D \in \mathrm{Cl}(X)$ for its linear equivalence class.

Definition 2.1 ([Kol18, Definition 15])**.** Let $\mathrm{Cl}(X)$ denote the group of divisors modulo linear equivalence, and let $\mathrm{Cl}^0(X)$ be the subgroup of divisors that are algebraically equivalent to 0. The *Néron–Severi class group* of X is the quotient $\mathrm{Cl}^{\mathrm{ns}}(X) = \mathrm{Cl}(X)/\mathrm{Cl}^0(X)$.

The natural inclusions $\mathrm{Pic}^0(X) \subset \mathrm{Cl}^0(X)$ and $\mathrm{NS}(X) \subset \mathrm{Cl}^{\mathrm{ns}}(X)$ are both isomorphisms if and only if every Weil divisor is Cartier (for example if X is locally factorial).

The class group can also be given a scheme structure via the Albanese variety. Note that here, following [Kol20], we use Albanese variety to refer to the classical, pre-Grothendieck notion of the Albanese variety [Wei50]. Since we work over an algebraically closed field, we will only consider the case when X has a smooth k-point: then the Albanese variety is the target of the universal rational map $\mathrm{alb}_X : X \dashrightarrow \mathbf{Alb}(X)$ from X to an abelian variety.

Theorem 2.2 ([Nér52], [LN59], [Kol18, Theorem 17], [Kol20])**.** *Let X be a proper, normal variety over an algebraically closed field k. Then*

$$\mathbf{Cl}^0(X) = \mathbf{Alb}(X)^*$$

where $\mathbf{Alb}(X)^*$ is the dual of the Albanese variety. The natural injection $\mathrm{Cl}^0(X) \to \mathbf{Cl}^0(X)(k)$ is an isomorphism, and $\mathrm{Cl}^{\mathrm{ns}}(X)$ is a finitely-generated abelian group.

In the literature, $\mathbf{Cl}^0$ is sometimes called the Picard variety [Wei50]. We refer the reader to [Kol20] for a modern treatment and to [Kle05] for more on the history.

Lemma 2.3 ([Kol18, Lemma 18])**.** *Let $\beta\colon X' \to X$ be a proper birational morphism between normal varieties, with exceptional divisors E_i. Then β_* induces isomorphisms*

$$\mathrm{Cl}^0(X') \cong \mathrm{Cl}^0(X),$$
$$\mathrm{Cl}^{\mathrm{ns}}(X')/\bigoplus_i \mathbb{Z}[E_i] \cong \mathrm{Cl}^{\mathrm{ns}}(X).$$

Base changing to a larger algebraically closed field may enlarge Cl^0 but will not affect $\mathrm{Cl}^{\mathrm{ns}}$.

Lemma 2.4. *Let X be a normal variety over an algebraically closed field k. Then $\mathrm{Cl}^{\mathrm{ns}}(X_{\mathbb{K}}) \cong \mathrm{Cl}^{\mathrm{ns}}(X)$ for any algebraically closed field $\mathbb{K}$ containing k.*

Proof. By [Kol20, 124.3] there is a proper normal variety X' that is birational to X and such that $\mathrm{Cl}^0(X') = \mathrm{Pic}^0(X')$, and $\mathrm{NS}(X'_{\mathbb{K}}) \cong \mathrm{NS}(X')$ by [MP12, Proposition 3.1]. □

2.2. **Very general.** Let X be a variety defined over an algebraically closed field k. A general choice will mean one made outside a finite union of proper closed k-subvarieties of the parameter space. When we use the term "very general" without any further specification, we refer to a choice made in a nonempty subset given by the complement of a countable union of proper closed k-subvarieties. If k is uncountable then the complement of such a countable union automatically contains a k-point.

In the latter section of this article, in order to make our very general choices more concrete, we will assume our algebraically closed field $\mathbb{K}$ has infinite transcendence degree, fix a field of definition, and make choices very general relative to this fixed field. In Weil's terminology $\mathbb{K}$ is a universal domain [Wei46, Chapter I, §1]. Although in practice the only countable fields this adds are the algebraic closures of $\mathbb{Q}(t_i \mid i \in \mathbb{N})$ and $\mathbb{F}_p(t_i \mid i \in \mathbb{N})$, it clarifies some instances when we need to make simultaneously very general choices.

Definition 2.5 ([Wei46, Chapter IV, §1; Chapter IX, §6])**.** Let $\mathbb{K}$ be a field and X a variety over $\mathbb{K}$. Then X can be defined over a field that is finitely generated over the prime subfield, and when we say a *field of definition* k_0 for X, we mean an algebraically closed field $k_0 \subset \mathbb{K}$ over which X is defined and such that k_0 has finite transcendence degree over the prime subfield ($\mathbb{Q}$ or $\mathbb{F}_p$). Note that k_0 is algebraic over a finitely-generated field and in particular is countable.

The Chow variety of X is defined over k_0. A subvariety Y of X is given by a $\mathbb{K}$-point of $\mathrm{Chow}(X)$, and Y is said to be defined over a subfield $k_0 \subset k' \subset \mathbb{K}$ if it is of the form $Y = Y' \otimes_{k'} \mathbb{K}$ for a subscheme Y' of $X \otimes_{k_0} k'$. There is a unique smallest field of definition k^Y of Y in X, which is the residue field of the morphism to $\operatorname{Spec} \mathbb{K} \to \mathrm{Chow}(X)$ giving Y [Kol17, 3.18.1 and Definition 3.21]. Note that k^Y is finitely generated and is generally not algebraically closed. If the coordinates of the $\mathbb{K}$-point $[Y]$ do not satisfy any algebraic equation over k_0 (or equivalently if the transcendence degree of k^Y over k_0 is equal to the dimension of the component of the Chow variety containing $[Y]$), we will say that Y is *very general over* k_0. If $\mathbb{K}$ is assumed to have infinite transcendence degree over the prime subfield then we can ensure that such Y exist.

We will say that $Y_1, \ldots, Y_r$ are *very general over* k_0 if each Y_i is very general over k_0 and the compositum of the fields k^{Y_j} for $j \neq i$. In Weil's language, the fields $k^{Y_1}, \ldots, k^{Y_r}$ are "independent" or "free with respect to each other" [Wei46, Chapter I, §2].

Very general over k_0 is more general than general, as it avoids any finite union of proper k_0-subvarieties.

Let X be a variety over a field $\mathbb{K}$ of infinite transcendence degree, and let $\mathcal{Y} \to B$ be a family of complete intersections in X. For $b \in B(\mathbb{K})$ let Y_b be the corresponding fiber. If we show that the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(Y_b)$ is surjective up to torsion whenever b is very general over a field of definition for X, then we will have shown that the restriction map to the class group of the geometric generic member $\mathcal{Y}_{\overline{\eta}}$ of the family is surjective up to torsion. Over an uncountable field, this is equivalent to the result for very general fibers, i.e. outside of a countable union of proper subvarieties of B, see [RS09, Section 3].

2.3. **Thin sets and Néron's theorem.** We briefly review some relevant facts about Néron's specialization theorem that we will use in Section 3.

Definition 2.6 ([Ser92, Chapter 3], [FJ08, Chapter 12])**.** Let k_1 be a field that is finitely generated over its prime subfield. A subset $T \subset \mathbb{P}^m_{k_1}(k_1)$ is *thin* if $T = f(X(k_1))$ for some k_1-variety X and morphism $f\colon X \to \mathbb{P}^m_{k_1}$ that is separable and generically finite onto its image and admits no rational section.

Every thin subset of $\mathbb{P}^m_{k_1}(k_1)$ is contained in a finite union of the following two types of thin sets:

(1) Thin subsets contained in proper subvarieties, and
(2) $f(X(k_1))$ for some separable dominant morphism $f\colon X \to \mathbb{P}^m_{k_1}$ with $\dim X = m$ and $\deg(f) \geq 2$.

If k_1 is a finitely-generated field not contained in $\overline{\mathbb{F}}_p$, then $\mathbb{P}^m_{k_1}(k_1)$ is not thin for $m \geq 1$ [Ser92, Theorem 3.4.1] [FJ08, Theorem 12.10]. Following [Kol20], for an arbitrary field k we say that $T \subset \mathbb{P}^m_k(k)$ is *field-locally thin* if $T \cap \mathbb{P}^m(k_1)$ is thin for every finitely-generated subfield $k_1 \subset k$.

Theorem 2.7 ([Nér52, Theorem 6])**.** *Let k be a field, $U \subset \mathbb{P}^m_k$ an open subset, and $\mathcal{A} \to U$ an abelian scheme. Assume the Mordell–Weil group $\mathcal{A}_\eta(K)$ is finitely generated, where K is the function field of $\mathbb{P}^m_k$ and $\mathcal{A}_\eta$ is the generic fiber of $\mathcal{A}$.*

Then there exists a subset $N \subset U(k)$ containing the complement of a field-locally thin set and such that the specialization map

$$\mathcal{A}_\eta(K) \longrightarrow \mathcal{A}_b(k)$$

is injective for $b \in N$.

2.4. **Bertini theorems.** In this section we recall some Bertini theorems for very ample divisors. We also show that if a line bundle is sufficiently positive, then the reducible locus is small (Lemma 2.13) and the sectional genus is large (Lemma 2.15).

Theorem 2.8 ([FOV99, Theorem 3.4.10])**.** *Let X be a variety over an infinite field k, D a Cartier divisor on X, and $|V| \subset |D|$ a finite-dimensional linear system. If*

(1) $|V|$ is not composed with a pencil (meaning the image of X under the morphism $|V|$ defines has dimension ≥ 2), and
(2) $\operatorname{codim}_X \operatorname{Bs}(|V|) \geq 2$,

then a general member of $|V|$ is irreducible.

Lemma 2.9 ([FOV99, Theorem 3.4.14])**.** *Let $X \subset \mathbb{P}^N_k$ be a projective scheme over an infinite field k. If X is regular (resp. normal, reduced, regular in codimension c), then for a general hyperplane $H \subset \mathbb{P}^N_k$, the intersection $H \cap X$ has the same property.*

Lemma 2.10 ([GK19, Corollary 3.4])**.** *Let k be an infinite field, X a normal equidimensional quasi-projective k-scheme of dimension ≥ 1, and $X \subset \mathbb{P}^N_k$ a locally closed embedding. Then a general hypersurface $H \subset \mathbb{P}^N_k$ satisfies*

(1) $X \cap H$ *is normal, and*
(2) $X^{\text{reg}} \cap H$ *is regular.*

Lemma 2.11 (Grothendieck's connectedness lemma [Gro05, Exposé XIII, Theorem 2.1])**.** *Let* (x, X) *be a local, excellent,* S_2 *scheme of pure dimension* ≥ 3 *and* $x \in D := (t = 0) \subset X$ *a Cartier divisor. Then* $D \setminus \{x\}$ *is connected.*

In the proof of Theorem 1.1 we want to work with linear systems where the locus of reducible members has codimension ≥ 2. This can sometimes fail, as in the case of $|\mathcal{O}_{\mathbb{P}^2}(2)|$ where the closure of the reducible plane conics is a divisor, but Lemma 2.13 will ensure these examples doesn't happen if the linear system is sufficiently positive.

Lemma 2.12. *Let* X *be a variety,* $\mathcal{L}$ *a line bundle and* $\Gamma \subset H^0(X, \mathcal{L})$ *a nonempty subspace, and* $\mathcal{B}$ *a line bundle defining a birational map. Then the linear system* $|\Gamma + \mathcal{B}| = \{r \otimes s \mid r \in \Gamma, b \in H^0(X, \mathcal{B})\} \subset H^0(X, \mathcal{L} \otimes \mathcal{B})$ *defines a birational map.*

Proof. Let $s_0, \ldots, s_N$ be a basis of $H^0(X, \mathcal{B})$ and $r_0, \ldots, r_m$ a basis of Γ. Then $\{r_i \otimes s_j \mid 0 \leq i \leq m, 0 \leq j \leq N\}$ is a basis of $|\Gamma| + |\mathcal{B}|$ and so the rational map it defines is given (up to isomorphism) by

$$\varphi \colon x \mapsto [r_0 s_0(x) : r_0 s_1(x) : \cdots : r_0 s_N(x) : r_1 s_0(x) : \cdots : r_m s_N(x)] \in \mathbb{P}^{(m+1)(N+1)}.$$

Denote the coordinates of $\mathbb{P}^{(m+1)(N+1)}$ by $[y_{00} : y_{01} : \cdots : y_{mN}]$. For each i, define $\pi_i \colon \mathbb{P}^{(m+1)(N+1)} \dashrightarrow \mathbb{P}^N$ to be the projection onto the coordinates $[y_{i0} : y_{i1} : \cdots : y_{iN}]$. Then $\pi_i \circ \varphi$ agrees with the rational map defined by $|\mathcal{B}|$ (away from $(s_i = 0)$) so its image has dimension n. So $\dim \varphi(X) = n$. $\square$

Lemma 2.13. *Let* X *be a variety of dimension* $n \geq 2$ *over an algebraically closed field* k*, and assume either*
(1) X *is normal, or*
(2) $\dim \operatorname{Sing}(X) = 0$.

Let $\mathcal{L}_0$ *be a very ample line bundle on* X*. Then the set of reducible divisors in* $|\mathcal{L}_0^d|$ *is contained in a closed subscheme of* $|\mathcal{L}_0^d|$ *of codimension* ≥ 2 *for*
(1) $d \geq 2$ *if the pair* $(X, \mathcal{L}_0)$ *is not* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$*, and*
(2) $d \geq 3$ *in general.*

Proof. The case where $X = \mathbb{P}^n$ and $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^n}(1)$ follows from an explicit computation, so we assume that $(X, \mathcal{L}_0) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. For each smooth (closed) point $x \in X$ and line bundle $\mathcal{M}$ on X, define $|B_x^{\mathcal{M}}| = \{D \in |\mathcal{M}| \mid x \in \operatorname{Sing} D\}$.

First we show that a general member of the linear system $|B_x^{\mathcal{L}_0^d}|$ is irreducible. Since $(X, \mathcal{L}_0) \neq (\mathbb{P}^n, \mathcal{O}(1))$, the set $|B_x^{\mathcal{L}_0}| = \{H \cap X \mid H \in |\mathcal{O}_{\mathbb{P}^{h^0(X,\mathcal{L}_0)-1}}(1)| \text{ and } T_{\varphi_{|\mathcal{L}_0|}(X),x} \subset H\}$ is nonempty, so by Lemma 2.12 the linear system $|B_x^{\mathcal{L}_0} + \mathcal{L}_0^{d-1}|$ defines a birational morphism to its image. Since $|B_x^{\mathcal{L}_0}| + |\mathcal{L}_0^{d-1}| \subset |B_x^{\mathcal{L}_0^d}|$, the span $|B_x^{\mathcal{L}_0} + \mathcal{L}_0^{d-1}|$ of these reducible divisors is also contained in $|B_x^{\mathcal{L}_0^d}|$ and so the morphism $|B_x^{\mathcal{L}_0^d}|$ defines is birational to its image. In particular its image has dimension $n \geq 2$. Since the base locus of $|B_x^{\mathcal{L}_0^d}|$ is x, which has codimension $n \geq 2$ in X, a general member of this linear system is irreducible by Theorem 2.8.

Replacing X by its image under the closed embedding $\varphi_{|\mathcal{L}_0^d|}$ defined by $\mathcal{L}_0^d$, we may assume that $\mathcal{L}_0^d = \mathcal{O}_{\mathbb{P}^N}(1)|_X$, where $\mathbb{P}^N = |\mathcal{L}_0^d|$, and that X is not contained in a hyperplane. Let $\mathcal{C} = \overline{\{(x, H) \mid x \in X^{\text{reg}} \text{ and } T_{X,x} \subset H\}} \subset X \times (\mathbb{P}^N)^*$ be the conormal variety of X, with projections $\pi_1 \colon X \times (\mathbb{P}^N)^* \to X$ and $\pi_2 \colon X \times (\mathbb{P}^N)^* \to (\mathbb{P}^N)^*$. The dual variety $\pi_2(\mathcal{C})$ is irreducible since $\mathcal{C}$ is irreducible of dimension $N - 1$.

Now consider the locus $S_{\text{reducible}}$ of hyperplanes whose intersection with X is reducible. This is a constructible subset of the closed subvariety of $|\mathcal{L}_0^d|$ parametrizing hyperplanes not regular in codimension 0 [FOV99, Theorem 3.3.14], and $S_{\text{reducible}} \subsetneq |\mathcal{L}_0^d|$ by Theorem 2.8. Note that by the Enriques–Severi–Zariski lemma, if $H \cap X$ is reducible then it is singular. So if X is smooth then $\overline{S_{\text{reducible}}} \subset \pi_2(\mathcal{C})$, and this containment is proper by the above discussion, so $\overline{S_{\text{reducible}}}$ has codimension ≥ 2 in $|\mathcal{L}_0^d|$.

If X is not smooth, suppose first that $\operatorname{Sing}(X)$ has dimension 0. For a closed point $x \in \operatorname{Sing}(X)$ consider the linear system $|\mathcal{L}_{0x}|$ of divisors passing through x. By Lemma 2.12 $|\mathcal{L}_{0x} + \mathcal{L}_0^{d-1}|$ is not composed of a pencil, and hence neither is $|\mathcal{L}_{0x}^d|$. By Theorem 2.8 a general member of $|\mathcal{L}_{0x}^d|$ is reducible, and so the reducible members R_x of $|\mathcal{L}_{0x}^d|$ form a closed subset of $\pi_1^{-1}(x)$ of dimension $\leq N - 2$.

Let $H \in S_{\text{reducible}}$. If the intersection of two components of $H \cap X$ contains a smooth point of X, then $H \in \pi_2(\mathcal{C})$. If not, then let $x \in \operatorname{Sing}(X)$ be a closed point in the intersection of two components of $H \cap X$. Then $H \in \pi_2(R_x)$. Since $\operatorname{Sing}(X)$ is a finite set, $\bigcup_{x \in \operatorname{Sing}(X)} B_x$ has dimension at most $N - 2$, so its image in $(\mathbb{P}^N)^*$ has dimension at most $N - 2$. Thus, in the case where $\dim \operatorname{Sing}(X) = 0$ we have shown that $\operatorname{codim}_{|\mathcal{L}_0^d|}(\overline{S_{\text{reducible}}}) \geq 2$.

Now assume X is normal. If $n = 2$ then $\dim \operatorname{Sing}(X) \leq 0$, so we may assume that $n \geq 3$. Then $S_{\text{reducible}}$ is the union of the two sets

(1) $\{H \mid H \cap X \text{ is reducible and } H \supset T_{X,x} \text{ for some closed } x \in X^{\text{reg}}\}$ and
(2) $\{H \mid H \cap X \text{ is reducible and } \operatorname{Sing}(H \cap X) \subset \operatorname{Sing}(X)\}$.

The closure of the first set has codimension ≥ 2 in $|\mathcal{L}_0^d|$ by the argument in the smooth case, so it remains to consider the second set.

For such an H, denote the irreducible components of $H \cap X$ by $Z_1, \ldots, Z_e$. Each intersection $Z_i \cap Z_j \subset \operatorname{Sing}(H \cap X)$ for $i \neq j$. For each codimension 3 point $x \in X$ contained in $Z_i \cap Z_j$, $\operatorname{Spec} \mathcal{O}_{X,x} \setminus \{x\}$ is connected by Lemma 2.11, so $Z_i \cap Z_j$ has codimension 2 in X. Since X is regular in codimension 1, and $Z_i \cap Z_j \subset \operatorname{Sing}(X)$ both have codimension 2 in X, we have that $Z_i \cap Z_j$ is a component of $\operatorname{Sing}(X)$. So $H \cap X$ must contain an $(n-2)$-dimensional irreducible component of $\operatorname{Sing}(X)$. By assumption $n - 2 \geq 1$, so requiring that H contains a irreducible component of $\mathrm{Sing}(X)$ imposes a codimension ≥ 2 condition on the elements of $|\mathcal{L}_0^d|$. So again in this case $\overline{S_{\text{reducible}}}$ has codimension ≥ 2 has codimension ≥ 2 in $|\mathcal{L}_0^d|$. ☐

Remark. If $n \geq 3$ then Lemma 2.13 holds for $d = 1$ as well without the normality assumption [NS52, Lemme 3] [Wei54, Lemme 4]. For surfaces in characteristic 0, the Kronecker–Castelnuovo theorem [Cas94] says that the general tangent hyperplane section of X is reducible, then X is either a ruled surface or the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Castelnuovo's proof uses properties in differential geometry that do not hold in positive characteristic.

Corollary 2.14. *Let $X \neq \mathbb{P}^2$ be a variety of dimension $n \geq 2$ over an algebraically closed field k, and assume that either*

(1) X is normal, or
(2) $\dim \mathrm{Sing}(X) = 0$.

Fix $r \leq n-1$, $\mathcal{L}_0$ a very ample line bundle on X, and $d \geq 2$ an integer. Then the locus of reducible complete intersections of r members of $|\mathcal{L}_0^d|$, is contained in a closed subset of $|\mathcal{L}_0^d|$ of codimension ≥ 2.

Proof. The result is clear if the pair $(X, \mathcal{L}_0^d)$ is $(\mathbb{P}^n, \mathcal{O}(1))$, so we assume otherwise. Induct on r: the $r = 1$ case is Lemma 2.13. If $r \geq 2$, let $\pi_1 \colon |\mathcal{L}_0^d| \times \cdots \times |\mathcal{L}_0^d| \to |\mathcal{L}_0^d|$ denote the first projection from the r-fold product, and let $R \subset |\mathcal{L}_0^d| \times \cdots |\mathcal{L}_0^d|$ denote the set of r-tuples $(H_1, \ldots, H_r)$ such that $H_1 \cap \cdots \cap H_r$ is reducible. First assume X is normal. For each fixed $H_1 \in |\mathcal{L}_0^d|$, we have that $\mathrm{codim}_{\pi_1^{-1}(H_1)}(R \cap \pi_1^{-1}(H_1)) \geq 2$ if H_1 is normal by hypothesis, and $\mathrm{codim}_{\pi_1^{-1}(H_1)}(R \cap \pi_1^{-1}(H_1)) \geq 1$ if H_1 is not normal. In the latter case, the locus of non-normal H_1 is contained in a codimension 1 subset of $|\mathcal{L}_0^d|$ by Bertini's theorem (Lemma 2.10). So by dimension counting we have that $\mathrm{codim}_{|\mathcal{L}_0^d| \times \cdots \times |\mathcal{L}_0^d|}(R) \geq 2$. The $\dim \mathrm{Sing}(X) = 0$ case follows by the same argument. ☐

We will need the following result on sectional genus to apply Lemma 4.5 in the proofs of Propositions 4.7 and 4.9.

Lemma 2.15. *Let X be a normal projective variety of dimension $n \geq 2$ over an algebraically closed field k and $\mathcal{L}_0$ an ample and basepoint-free line bundle on X. For any smooth complete intersection curve D of members of $|\mathcal{L}_0^d|$ such that D is contained in the smooth locus of X, the cokernel of the restriction map*

$$\mathbf{Cl}^0(X) \longrightarrow \mathbf{Jac}(D)$$

has positive dimension if

(1) $d \geq 2$, unless $X = \mathbb{P}^2$ and $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}(1)$, and
(2) $d \geq 3$ in general.

Proof. Let $C \subset X^{\text{reg}}$ be a smooth complete intersection curve of members of $|\mathcal{L}_0|$. Since the morphism $\mathbf{Jac}(C) \to \mathbf{Alb}(X)$ is surjective [Kol20, Lemma

130] we have $p_a(C) \geq \dim \mathbf{Alb}(X) = \dim \mathbf{Cl}^0(X)$. So we need to show that $p_a(D) > p_a(C)$ for D in the statement of the lemma.

First suppose X is a surface. The arithmetic genus of any member of $|\mathcal{L}_0^d|$ is equal to that of $\sum_{i=1}^d C_i$ for curves $C_i \in |\mathcal{L}_0|$, which is equal to

$$p_a(\sum_{i=1}^d C_i) = \sum_{i=1}^d p_a(C_i) + (1-d) + \frac{d(d-1)}{2}\mathcal{L}_0^2.$$

If $d = 2$ then $p_a(C_1 + C_2) = p_a(C_1) + p_a(C_2) + \mathcal{L}_0^2 - 1$, and this is strictly larger than $p_a(C_i)$ unless $p_a(C_i) = 0$ and $\mathcal{L}_0^2 = 1$. In this case, $X = \mathbb{P}^2$ and $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}^2}(1)$, and any plane curve of degree $d \geq 3$ has positive genus.

If $n \geq 3$, let D be a complete intersection curve of members of $|\mathcal{L}_0^d|$. Similarly one computes that

$$p_a(D) = dp_a(C) + 1 - d + d\frac{(d^{n-1}-1)(n-1)}{2}\mathcal{L}_0^n$$

where C is a complete intersection curve of members of $|\mathcal{L}_0|$. Since $n \geq 3$ and $\mathcal{L}_0$ is ample we have

$$p_a(D) - dp_a(C) = 1 - d + d\frac{(d^{n-1}-1)(n-1)}{2}\mathcal{L}_0^n > d^n - 2d + 1,$$

which is positive for $d \geq 2$.

□

In order to ensure that the conclusions of Corollaries 2.14 and 2.15 hold, throughout this article we will assume $\mathcal{L} = \mathcal{L}_0^d$ for some very ample line bundle $\mathcal{L}_0$ on X and integer $d \geq 2$.

3. Injectivity

In this section we prove injectivity of the restriction map on class groups.

Proposition 3.1 (Corollary 3.7)**.** *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field k, let $\mathcal{L}_0$ be a very ample line bundle, and let $d \geq 2$ be an integer. Then the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

is injective for a general complete intersection Y of at most $n-2$ members of $|\mathcal{L}_0^d|$.

Weil proved injectivity of the restriction map on Cl to when Y is very general over k for $n \geq 3$ and $d \geq 1$ (and more generally for X and $\mathcal{L}_0^d$ such that the reducible locus in $|\mathcal{L}_0^d|$ has codimension ≥ 2) by considering a pencil in $|\mathcal{L}_0^d|$ [Wei54, Théorème 2]. We will fiber X by curves and use Néron's specialization theorem to show the result for general members.

3.1. **Sections of the relative Jacobian.** We will use the relative Jacobian to understand divisors. If $f\colon X \to B$ is a projective and flat morphism of relative dimension one, locally of finite presentation, and with geometrically reduced and irreducible fibers, then $\mathbf{Pic}_{X/B}$ is representable as a smooth separated B-scheme. If f is smooth over $U \subset B$, then $\mathbf{Pic}^0_{X|_U/U}$ is an abelian U-scheme [BLR90, Theorem 9.3.1, Proposition 9.4.4].

We now fix some notation and assumptions that will be used throughout the paper and describe the setting in which we will consider the relative Jacobian. Since the restriction map from $\mathrm{Pic}(\mathbb{P}^2)$ to any curve is injective, we may assume $X \neq \mathbb{P}^2$.

Assumption 1. *Let $X \neq \mathbb{P}^2$ be a normal variety of dimension $n \geq 2$ over an algebraically closed field k of arbitrary characteristic, $\mathcal{L}$ a very ample line bundle on X, and $|V| \subset |\mathcal{L}|$ a general linear system of dimension $n-1$. Assume that $\mathcal{L} = \mathcal{L}_0^d$ for an integer $d \geq 2$ and a very ample line bundle $\mathcal{L}_0$ on X. Fix a base point x of the linear system $|V|$. We also fix a field of definition $k_0 \subset k$ for X and $|V|$, recalling by our conventions in Definition 2.5 that this means k_0 is algebraically closed and has finite transcendence degree over the prime subfield.*

The linear system $|V|$ defines a rational map $\varphi\colon X \dashrightarrow \mathbb{P}^{n-1}$, which is a morphism away from the finite set $\mathrm{Bs}\,|V|$. Let $\varphi'\colon X' \to \mathbb{P}^{n-1}$ be the normalization of the closure of the graph of φ'; by Lemma 2.10 this is a smooth morphism over a dense open subset $U_{\mathrm{sm}} \subset \mathbb{P}^{n-1} \setminus \varphi'(\mathrm{Sing}(X')) \subset \mathbb{P}^{n-1}$. Let $\beta\colon X' \to X$ be the induced birational morphism. This is an isomorphism away from the base locus of $|V|$, which by generality is contained in the regular locus of X, so the pullback $\beta^*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(X')$ is defined. Let E_x be the exceptional divisor over x.

For a subvariety $V \subset X$ we denote by $V' \subset X'$ its strict transform. For a point $b \in \mathbb{P}^{n-1}$ the closure $C_b = \overline{\varphi^{-1}(b)}$ of the fiber of φ is a complete intersection curve of members of $|V|$, and $C'_b = \varphi'^{-1}(b)$. Since the conditions of Lemma 2.13 are satisfied, we have that the locus R of reducible fibers has codimension ≥ 2 in $|V|$. Let U_{i} denote the complement of the closure of R, let U_{sm} denote the locus of smooth fibers, and let C'_η denote the generic fiber of φ', which is a curve over the function field K of $\mathbb{P}^{n-1}$.

We consider the relative Jacobian of $\varphi'\colon X' \to \mathbb{P}^{n-1}$ in the above setting. Then

- $\mathbf{Pic}^0_{X'|_{U_{\mathrm{i}}}/U_{\mathrm{i}}}$ is a variety and smooth over U_{i}, and
- $\mathbf{Pic}^0_{X'|_{U_{\mathrm{sm}}}/U_{\mathrm{sm}}} \cong \mathbf{Pic}^0_{X'|_{U_{\mathrm{i}}}/U_{\mathrm{i}}} \times_{U_{\mathrm{i}}} U_{\mathrm{sm}}$ with is an abelian U_{sm}-scheme with zero section $b \mapsto \mathcal{O}_{C'_b}$. We frequently denote it by $\mathbf{Jac}(X'/\mathbb{P}^{n-1})$.

The section of φ' given by the exceptional divisor E_x defines a rational map $X' \dashrightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1})$ that is a morphism on U_{sm}, and for any $U \subset U_{\mathrm{sm}}$ we have an isomorphism

$$\mathrm{Pic}^0(X'|_U) \cong \mathbf{Pic}^0_{X'|_{U_{\mathrm{i}}}/U_{\mathrm{i}}}(U)$$

between equivalence classes of algebraically trivial line bundles and sections [BLR90, Proposition 8.1.4].

We will study divisors on X' by relating them to rational sections of the relative Jacobian [Nér52, Chapitre I, §8]. An algebraically trivial Cartier divisor on X' defines a section of $\mathbf{Pic}^0_{X'|_{U_\mathrm{i}}/U_\mathrm{i}} \to U_\mathrm{i}$ by pullback. An algebraically trivial Weil divisor D on X' defines a *rational* section σ_D of $\mathbf{Pic}^0_{X'|_{U_\mathrm{i}}/U_\mathrm{i}} \to U_\mathrm{i}$ over U_sm by restricting first to the regular locus of X' and then pulling back the Cartier divisor $\mathcal{O}_{X'^{\mathrm{reg}}}(D|_{X'^{\mathrm{reg}}})$. Note that this rational map does not in general extend over all of U_i.

By twisting down by an appropriate multiple of E_x, we can extend this to any Weil divisor on X':

Lemma 3.2. *The map*

$$\begin{aligned} \mathrm{Cl}(X') &\longrightarrow \mathbf{Jac}(C'_\eta)(K) = \{\textit{sections of } \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \to U_\mathrm{sm}\} \\ D' &\longmapsto (D' - \deg(D'|_{C'_\eta})E_x)|_{C'_\eta} \end{aligned} \tag{1}$$

is a surjection with kernel generated by E_x and $\varphi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$ for some $a \in \mathbb{Z}$.

Proof. The morphism φ' admits a section, so a section σ of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \to U_\mathrm{sm}$ is the same as a divisor class in $\mathrm{Pic}^0(X'|_{U_\mathrm{sm}})$ whose restriction to C'_b agrees with $\sigma(b) \in \mathrm{Jac}(C'_b)$. So surjectivity follows from surjectivity of $\mathrm{Cl}(X') \to \mathrm{Cl}(X'|_{U_\mathrm{sm}}) \cong \mathrm{Pic}(C'_\eta)$.

Next we consider the kernel. The codimension 1 irreducible components $\{\gamma_i\}_i$ of $\mathbb{P}^{n-1}\backslash U_\mathrm{sm}$ are degree a_i hypersurfaces in $\mathbb{P}^{n-1}$. By Lemma 2.14, each preimage $\varphi'^{-1}(\gamma_i)$ is irreducible, so the kernel is generated by $\varphi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$ where $a = \gcd\{a_i\}$. So for a divisor D' to be in the kernel of the map (1), it must satisfy $D' - \deg(D'|_{C'_\eta})E_x \in \langle \varphi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)\rangle$. Since $d \geq 2$ in Assumption 1 we have $\#\,\mathrm{Bs}\,|V| \geq 2$, so E_x is not the only exceptional divisor of β, and for any other exceptional divisor E' the restrictions $E_x|_{C'_\eta}$ and $E'|_{C'_\eta}$ are independent in $\mathrm{Pic}(C'_\eta)$. So the kernel of (1) is generated by E_x and $\varphi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$. □

The image of $\mathrm{Cl}(X)$ is disjoint from the subgroup of $\mathrm{Cl}(X')$ generated by E_x and $\varphi'^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$, so in particular we have:

Corollary 3.3. *The map $\mathrm{Cl}(X) \to \mathrm{Jac}(C'_\eta)$ induced by the homomorphism (1) of Lemma 3.2 is injective.*

Remark. For an abelian scheme $\mathcal{A} \to B \to \operatorname{Spec} k$ we denote by $\mathrm{Tr}_{B/k}(\mathcal{A})$ and $\mathrm{Im}_{B/k}(\mathcal{A})$ the Chow B/k-trace and Chow B/k-image, respectively. These are abelian k-varieties that depend only on the field extension K/k and the abelian variety $\mathcal{A} \times_B \operatorname{Spec} K$ over $K := k(B)$ [GS13, Remark 3.3]. We refer the reader to [Con06] and [GS13, Section 3] for more details.

Chow introduced the K/k-trace and K/k-image and used this theory to construct Cl^0 and the Albanese variety [Cho52, Cho55]. Specifically, there is a canonical isomorphism

$$\mathbf{Cl}^0(X') \cong \mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})).$$

In one direction, from the rational maps

$$X' \dashrightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \xrightarrow{\cong} \mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee \longrightarrow \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee) \times U_{\mathrm{sm}}$$

we get a map $X' \dashrightarrow \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee)$ inducing a homomorphism $\mathbf{Alb}(X') \to \mathrm{Im}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})^\vee)$. This dualizes to a homomorphism

$$\mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1})) \longrightarrow \mathbf{Cl}^0(X')\ .$$

Conversely, the map (1) defines a morphism $\mathbf{Cl}^0(X')\times U_{\mathrm{sm}} \to \mathbf{Jac}(X'/\mathbb{P}^{n-1})$ of abelian U_{sm}-schemes and hence a homomorphism of abelian varieties $\mathbf{Cl}^0(X') \to \mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1}))$ by the universal property of the Chow trace.

3.2. **The connected component** Cl^0**.** We first show that a stronger result holds for the subgroup of algebraically trivial divisors: the restriction map on Cl^0 is an isomorphism for a general divisor.

The following lemma is elementary and presumably well-known, but we include it here for completeness. It also holds if U is replaced by a quasi-finite, generically unramified cover from a normal variety.

Lemma 3.4. *Let k be an algebraically closed field and $\pi\colon \mathcal{A} \to U \subset \mathbb{P}^m_k$ an abelian scheme. Then $\mathrm{Tr}_{U/k}(\mathcal{A}) \cong \mathrm{Tr}_{L\cap U/k}(\mathcal{A}|_{L\cap U})$ for a general line $L \subset \mathbb{P}^m_k$.*

Proof. For any line $L \subset \mathbb{P}^m_k$, we can restrict the morphism $\mathrm{Tr}_{U/k}(\mathcal{A})\times U \to \mathcal{A}$ of abelian U-schemes to L, so $\mathrm{Tr}_{U/k}(\mathcal{A}) \to \mathrm{Tr}_{L\cap U/k}(\mathcal{A}|_{L\cap U})$ comes from the universal property. For a map in the opposite direction, pick $b \in B$ general and consider the family of lines $L_\lambda \ni b$. For each λ let $\mathrm{Tr}_{L_\lambda\cap U/k}(\mathcal{A}|_{L_\lambda\cap U})$ be the trace of the abelian scheme $\mathcal{A}_{L_\lambda\cap U} \to L_\lambda \cap U$. Then the fibers over b of the maximal abelian k-subschemes of $\mathcal{A}|_{L_\lambda\cap U} \to L_\lambda \cap U$

$$\{\tau_\lambda(\mathrm{Tr}_{L_\lambda\cap U/k}(\mathcal{A}|_{L_\lambda\cap U}) \times (L_\lambda \cap U))_b \mid L_\lambda \ni b\}$$

form a family of abelian subvarieties of $\mathcal{A}_b$. So for a general λ they must be constant [Cho55, §2] and equal to some abelian k-variety A_0, which is isogenous to $\mathrm{Tr}_{L_\lambda\cap U/k}(\mathcal{A}|_{L_\lambda\cap U}) \times (L_\lambda \cap U)$, so the traces for these λ are isogenous. Pick one and call it T. Then we have $T \otimes_k k(U) \to \mathcal{A}_\eta$, so by the universal property of the trace we get a morphism of abelian varieties $T \to \mathrm{Tr}_{U/k}(\mathcal{A})$. □

Corollary 3.5. *In the setting of Assumption 1, assume $n \geq 3$. Then for a general complete intersection Y of $\leq n-2$ members of $|\mathcal{L}|$, the restriction map $\mathrm{Cl}^0(X) \to \mathrm{Cl}^0(Y)$ is an isomorphism.*

Proof. First we consider complete intersection surfaces of members of $|V|$. Let $L \subset \mathbb{P}^{n-1}$ be a general line corresponding to a complete intersection surface $S \subset X$. Then the abelian schemes $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \to U_{\mathrm{sm}}$ and $\mathbf{Jac}(X'/\mathbb{P}^{n-1})|_{L\cap U_{\mathrm{sm}}} \to L \cap U_{\mathrm{sm}}$ have the same trace, and these are the same as $\mathrm{Cl}^0(X)$ and $\mathrm{Cl}^0(S)$, respectively, as noted in the remark below Corollary 3.3.

The isomorphism $\mathrm{Cl}^0(X) \cong \mathrm{Cl}^0(Y)$ for a general divisor Y in $|V|$ follows from applying the surface case to $S \hookrightarrow Y$ and $S \hookrightarrow X$. Since $|V| \subset |\mathcal{L}|$ is general, then repeatedly applying the divisor case gives the result. □

3.3. **Injectivity for** Cl**.** In the following lemma we assume $k \neq \overline{\mathbb{F}}_p$ to use Néron's specialization theorem.

Lemma 3.6. *If $k \neq \overline{\mathbb{F}}_p$, then for k-points b away from a field-locally thin set in $\mathbb{P}^{n-1}(k)$, the map*

$$\mathbf{Jac}(C'_\eta)(K) \longrightarrow \mathbf{Jac}(C'_b)(k)$$

is injective.

Proof. Let $A = \mathrm{Tr}_{U_{\mathrm{sm}}/k}(\mathbf{Jac}(X'/\mathbb{P}^{n-1}))$. After possibly replacing U_{sm} by a smaller open subset U, by [Con06, Theorem 6.4] and Poincaré reducibility we can find isogenies of abelian U-schemes

$$(A \times U) \times \mathcal{Q} \longrightarrow \mathbf{Jac}(X'/\mathbb{P}^{n-1}) \longrightarrow (A \times U) \times \mathcal{Q}$$

where $\mathcal{Q} = \mathbf{Jac}(X'/\mathbb{P}^{n-1})/(A \times U)$ is the quotient and $\mathrm{Tr}_{U/k}(\mathcal{Q}) = 0$. So it suffices to consider separately the two cases of a constant abelian scheme and one with trivial trace.

For the traceless family $\mathcal{Q} \to U$, the group of rational sections is finitely generated by the Lang–Néron theorem [LN59, Theorem 1]. So by Néron's Theorem 2.7 there are infinitely many points $b \in U_{\mathrm{sm}}(k)$ outside a field-locally thin set such that the specialization $\mathcal{Q}_\eta(K) \to \mathcal{Q}_b(k)$ is injective. Next, the constant part has only constant sections, since a non-constant section would give a non-constant morphism $U \to A$, which cannot exist since U is an open subset of $\mathbb{P}^{n-1}$. A priori we could have torsion in the kernel, but this does not happen by Corollary 3.3 since $A = \mathrm{Cl}^0(X)$. □

Remark. If k has infinite transcendence degree over the prime subfield, Lemma 3.6 can be proven for b very general over k_0 without using the Lang–Néron theorem or Néron's specialization theorem. In this setting the group of rational sections of $\mathcal{Q} \to U$ is countable because each rational section gives an isolated point of the Chow variety [GS13, Lemma 3.6], so any b with coordinates transcendental over a field of definition for $\mathcal{Q}$ is outside the images of their intersections.

Putting this together with Corollary 3.3 shows injectivity of the restriction map from X to infinitely many curves. We can factor this through higher-dimensional complete intersections in $|V|$:

Corollary 3.7. *If $n \geq 3$ in Assumption 1, then without further assumptions on the ground field k, the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$ is injective for a general complete intersection Y of $\leq n-2$ members of $|\mathcal{L}|$.*

Proof. We first show injectivity of the restriction map for a general complete intersection surface S of members of $|V|$, since this will imply injectivity of the restriction map to any divisor in $|V|$ containing S. If $k = \overline{\mathbb{F}}_p$, let $k \subsetneq k'$ be an extension of algebraically closed fields. Let N be the set of k'-points for which the specialization map of Lemma 3.6 is injective; the complement of N is contained inside a field-locally thin subset of $\mathbb{P}^{n-1}(k')$. Since for any finitely-generated field $k_1 \subset k'$ the set $\mathbb{P}^{n-1}(k_1) \setminus N(k_1)$ is a finite union of type 1 and type 2 thin sets (Definition 2.6), and since $\mathbb{P}^1(k_1) \subset \mathbb{P}^1(k_1)$ is not thin, then for a general k-line $L \subset \mathbb{P}^{n-1}_k$, the base change $L_{k'}$ will contain a k'-point b in N and such that $b \in U_{\mathrm{sm}}(k')$. For such L and $b \in L_{k'}(k')$, the compositions of the maps in Corollary 3.3 and Lemma 3.6 is injective and factors through $\mathrm{Cl}(S_{k'})$, where $S = \overline{\varphi^{-1}(L)}$ is the complete intersection surface corresponding to the line L, so $\mathrm{Cl}(X_{k'}) \to \mathrm{Cl}(S_{k'})$ is injective. The base change to k' does not change $\mathrm{Cl}^{\mathrm{ns}}$ by Lemma 2.4, and we have $\mathrm{Cl}^0(X) = \mathrm{Cl}^0(S)$ and $\mathrm{Cl}^0(X_{k'}) = \mathrm{Cl}^0(S_{k'})$ by Corollary 3.5, so we conclude that over k the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(S)$ is injective.

Since $|V| \subset |\mathcal{L}|$ was general, this implies that the restriction map to a general divisor in $|\mathcal{L}|$ is injective. Repeatedly applying the divisor case gives the result for general complete intersections. □

4. Surjectivity

In this section we prove surjectivity modulo torsion of the restriction map on class groups.

Proposition 4.1 (Corollary 4.14, Corollary 4.4, Corollary 4.13)**.** *Let X be a normal variety of dimension $n \geq 3$ over an algebraically closed field $k \neq \overline{\mathbb{F}}_p$, and let $\mathcal{L}_0$ be a very ample line bundle on X. Then there exists $Y \in |\mathcal{L}_0^d|$ in the complement of a countable union of proper subvarieties such that the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

(1) has torsion cokernel for $d = 4$ or ≥ 6 if $n = 3$,
(2) is surjective for $d \geq 2$ if $n \geq 4$ and if k either
  (a) has characteristic 0 or
  (b) has infinite transcendence degree over its prime subfield, and
(3) has p-power torsion cokernel for $d \geq 2$ if $n \geq 4$ and $\mathrm{char}\, k = p$.

4.1. Surjectivity over fields of infinite transcendence degree.

In this section we assume the ground field $\mathbb{K}$ has infinite transcendence degree over its prime subfield and use the notion of very general over k_0 discussed in §2.2.

We will use the following result of Graber–Starr [GS13] on extending sections of abelian schemes. Their result is stated for very general incident line pairs and 2-planes over an uncountable field, but their proof works for fields of infinite transcendence degree.

Theorem 4.2 ([GS13, Theorem 1.3])**.** *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree, B a normal variety of dimension ≥ 2, and $f\colon B \to \mathbb{P}^{\dim B}$ a generically finite, generically unramified morphism. Let $\mathcal{A} \to B$ be an abelian scheme, and let $k_0 \subset \mathbb{K}$ be a field of definition for $\mathcal{A}$ and B. Then for any pair of incident lines $L_1 \cup L_2 \subset \mathbb{P}^{\dim B}$ very general over k_0, the restriction map*

$$\{\textit{sections of } \mathcal{A} \to B\} \to \{\textit{sections of } \mathcal{A}|_{f^{-1}(L_1 \cup L_2)} \to f^{-1}(L_1 \cup L_2)\}$$

is a bijection. The same result also holds with incident line pair replaced by 2-plane.

Applying their result for incident line pairs to the relative Jacobian shows surjectivity for a reducible member of $|\mathcal{L}^2|$:

Lemma 4.3. *In the setting of Assumption 1, assume that X is defined over a field $\mathbb{K}$ of infinite transcendence degree and that $n \geq 3$. Let S_1, S_2 be a pair of complete intersection surfaces of members of $|V|$ very general over k_0, and let D_i be a Weil divisor on each S_i such that $D_1|_{S_1 \cap S_2} \sim D_2|_{S_1 \cap S_2}$. Then there is a Weil divisor D on X, uniquely determined up to linear equivalence, such that $D|_{S_i} \sim D_i$ for each i. That is, the restriction map*

$$\mathrm{Cl}(X) \to \mathrm{Cl}(S_1) \times_{\mathrm{Pic}(S_1 \cap S_2)} \mathrm{Cl}(S_2)$$

is an isomorphism.

Proof. Let S_i' denote the strict transform of S_i, let L_i be the line such that $S_i' = \varphi'^{-1}(L_i)$, and consider the fiber product $\mathrm{Cl}(S_1') \times_{\mathrm{Pic}(S_1 \cap S_2)} \mathrm{Cl}(S_2')$, where the restriction maps $\mathrm{Cl}(S_i') \to \mathrm{Cl}(S_i) \to \mathrm{Pic}(S_1 \cap S_2)$ are pushforward by $\pi|_{S_i'}$ composed with the Gysin homomorphisms [Ful98, Chapter 6]. Let D_i be divisors on S_i whose restrictions agree in $\mathrm{Pic}(S_1 \cap S_2)$. The pullback $\beta|_{S_i'}^* D_i$ of D_i is defined because the base points of $|V|$ are smooth points of S_i by generality.

The restrictions $(\beta|_{S_i'}^* D_i)|_{C_\eta'}$ have the same degree, so by the procedure of Lemma 3.2 D_1 and D_2 define a section of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \to U_{\mathrm{sm}}$ over $(L_1 \cup L_2) \cap U_{\mathrm{sm}}$. This extends uniquely to a section σ of $\mathbf{Jac}(X'/\mathbb{P}^{n-1}) \to U_{\mathrm{sm}}$ by Theorem 4.2 [GS13, Theorem 1.3]. By Lemma 3.2 we can find a divisor D' on X' defining σ and such that $(\pi_* D')|_{S_i} \sim D_i$ for each i. □

The same argument applied to the 2-plane case of Theorem 4.2 gives surjectivity for complete intersection threefolds of members of $|V|$ that are very general over k_0. We also get surjectivity to higher-dimensional complete intersections by factoring the restriction map.

Corollary 4.4. *In the setting of Assumption 1, assume that X is defined over a field $\mathbb{K}$ of infinite transcendence degree and that $n \geq 4$. For a complete intersection Y of $r \leq n-3$ members of $|\mathcal{L}|$ very general over k_0, the* *restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y)$$

is surjective.

Proof. It suffices to show the case when $Y \in |V|$ is a divisor. If $\dim Y \geq 4$, let Z be a threefold obtained as the complete intersection of Y with members of $|V|$ very general over k_0. The restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(Z)$ is surjective and factors through $\mathrm{Cl}(Y) \to \mathrm{Cl}(Z)$, which is injective by Corollary 3.7. □

In order to specialize from a sufficiently general member of $|\mathcal{L}^2|$ to a reducible one in $|\mathcal{L}| + |\mathcal{L}|$, we need the following conjecture of Kollár on independence of intersection points. Voisin proved the conjecture over fields of infinite transcendence degree, and we include her proof below. This result will ensure in Proposition 4.7 that the points p_i are independent in $\mathrm{Pic}(C)$ modulo the hyperplane section, as in the case for quartic surfaces in $\mathbb{P}^4$ when C was a degree 4 elliptic curve in §1.2.

Lemma 4.5 ([Voi20]). *Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree. Let C be a smooth projective curve over $\mathbb{K}$ and $\mathcal{M}$ a very ample line bundle on C. Assume that either $\operatorname{char} \mathbb{K} = 0$ or that $\mathcal{M} = \mathcal{M}_0^{\otimes d}$ for an integer $d \geq 2$ and $\mathcal{M}_0$ a very ample line bundle. Let $A \subsetneq \mathbf{Pic}^0(C)$ be an abelian subvariety and $\mathcal{M} \in G \subset \mathrm{Pic}(C)$ a finitely-generated subgroup. Let $k^C \subset \mathbb{K}$ be a field of definition for C, $\mathcal{M}$, A, and G.*

For a divisor $D \in |\mathcal{M}|$ write $\operatorname{Supp} D = \bigcup_i p_i(D)$. Then for $D \in |\mathcal{M}|$ very general over k^C, the map

$$\bigoplus_i \mathbb{Z}[p_i(D)] / \sum_i [p_i(D)] \hookrightarrow \mathrm{Pic}(C)/\langle A(\mathbb{K}), G\rangle$$

is an injection.

Proof of Voisin. Let $d = \deg \mathcal{M}$, let $Z \subset C \times |\mathcal{M}|$ be the universal family, and let $U \subset |\mathcal{M}|$ be the open subset parametrizing divisors consisting of d distinct points. Let $Z^{(d)} \to |\mathcal{M}|$ be the universal family of divisors in $|\mathcal{M}|$ with an ordering of the d (not necessarily distinct) points.

First we show that $Z_U^{(d)}$ is irreducible, or equivalently that the monodromy group of $Z_U \to U$ is the full symmetric group S_d. It suffices to show that the monodromy group is 2-transitive and contains a transposition. 2-transitivity is equivalent to irreducibility of $Z_U^{(2)}$, which can be identified with the set $\{(p_1, p_2, D) \mid p_1, p_2 \in \operatorname{Supp} D\} \subset C \times C \times U$. The fiber over a point $(p_1, p_2) \in C \times C$ is the linear system $|\mathcal{M} - p_1 - p_2|$, so since $\mathcal{M}$ is very ample the dimension of this linear system is independent of the p_i and $Z_U^{(2)} \to U$ is a projective bundle and thus irreducible. Deforming a section with a double zero gives a transposition.

Next, for $\gamma \in G$ and integers $n_1, \dots, n_d, m \in \mathbb{Z}$ such that $m \neq 0$ and $(n_1, \dots, n_d) \neq (0, \dots, 0)$, define the subset

$$Z(n_1, \dots, n_d, m, \gamma) = \left\{ (p_1, \dots, p_d, \sum_{i=1}^{d} p_i) \in Z_U^{(d)} \mid m(\gamma - \sum_{i=1}^{d} n_i[p_i]) \in A \right\}$$

of $Z_U^{(d)} \subset C \times \cdots \times C \times |\mathcal{M}|$. Then

$$\bigcup_{(n_1, \dots, n_d, m, \gamma)} Z(n_1, \dots, n_d, m, \gamma) \subset Z_U^{(d)}$$

is a countable union of k^C-subvarieties, and to show that the complement contains $\mathbb{K}$-points, it suffices to show that each $Z(n_1, \dots, n_d, m, \gamma) \subsetneq Z_U^{(d)}$.

By contradiction, assume that some $Z(n_1, \dots, n_d, m, \gamma) = Z_U^{(d)}$. Then $m(\gamma - \sum_{i=1}^{d} n_i[p_i]) \in A$ for every $(p_1, \dots, p_d, \sum_{i=1}^{d} p_i) \in Z_U^{(d)}$, and since the monodromy group is the entire symmetric group S_d, this also holds for any permutation of the p_i. Now the permutation representation on $\mathbb{Q}^d$ is the sum of two irreducible representations: the diagonal and its complement, which is spanned by differences of the basis vectors. So we have that either $n_1 = \cdots = n_d$, or that $m(\gamma - n[p_1] + n[p_2]) \in A$ for any $p_1, p_2 \in C$. In the second case we get that $\mathbf{Pic}^0(C)/A$ is torsion since the differences $[p_1] - [p_2]$ generate Pic^0, which is a contradiction. So we are in the first case, i.e. a multiple of $\sum_{i=1}^{d} p_i$. □

We will also need to replace X by suitable alteration in the proof of Proposition 4.7 to use an argument involving specialization of $\mathbb{Q}$-Cartier divisors.

Lemma 4.6. *Let X be a normal threefold over an algebraically closed field k, $\mathcal{L}$ a very ample line bundle on X, and $|\Lambda| \subset |\mathcal{L}|$ a general pencil. Then there exists a morphism from a projective variety $\psi\colon X^+ \to X$ such that*

(1) $\psi\colon X^+ \to X$ is finite and purely inseparable over the regular locus of X,
(2) X^+ is $\mathbb{Q}$-factorial except possibly over finitely many points $q_1, \dots, q_r$ of X,
(3) the pullback of a general member of $|\Lambda|$ to X^+ is smooth, and
(4) for any $b \geq 1$ the pullback of a general member of $|\mathcal{L}^b|$ is smooth along the exceptional divisors of $\psi\colon X^+ \to X$. In particular, the pullback of a general member of $|\mathcal{L}^b|$ is $\mathbb{Q}$-factorial.

Proof. If $\operatorname{char} k = 0$, then any resolution of singularities $\psi\colon X^+ \to X$ works, so assume $\operatorname{char} k = p > 0$. Let $\mathcal{Y} \to |\Lambda|$ be the total space of the pencil. The generic fiber Y is a geometrically normal variety over the function field K of $|\Lambda| \cong \mathbb{P}^1$, and after a finite purely inseparable field extension the base change $Y \otimes_K K^{1/p^e}$ admits a resolution of singularities Y^+ that is smooth over K^{1/p^e}. Spreading out Y^+ and $Y \otimes_K K^{1/p^e}$ over an open subset U^{1/p^e} of

$|\Lambda|^{1/p^e}$ we get a smooth family $\mathcal{Y}^+ \to U^{1/p^e}$ of surfaces, a family $\mathcal{Y}|_U \times_U U^{1/p^e}$ of normal surfaces, and a diagram

$$\begin{array}{ccccccc} \mathcal{Y}^+ & \xrightarrow{\text{bir}} & \mathcal{Y}|_U \times_U U^{1/p^e} & \xrightarrow{\text{univ homeo}} & \mathcal{Y}|_U & \xrightarrow[\mathrm{Bl}_{\mathrm{Bs}\,|\Lambda|}]{\text{bir}} & \pi_1(\mathcal{Y}|_U) \subset X \\ & & \downarrow & & \downarrow & & \\ & & U^{1/p^e} & \longrightarrow & U & \dashleftarrow & \end{array}$$

where $\pi_1(\mathcal{Y}|_U)$ is a dense open subset of X whose complement is the union of finitely many divisors $Y_1, \dots, Y_l$ in $|\Lambda|$. By Bertini's theorem (after possibly removing finitely many points from U) the birational morphism $\mathcal{Y}^+ \to \mathcal{Y}|_U \times_U U^{1/p^e}$ is an isomorphism over the regular locus of X.

Let $\theta\colon Z \to X$ be the normalization of X in K^{1/p^e}. Then $\theta^{-1}(X^{\text{reg}}) \to X^{\text{reg}}$ is a purely inseparable cover that agrees with $\mathcal{Y}^+$ over the intersection $X^{\text{reg}} \cap \pi_1(\mathcal{Y}|_U)$, so we may glue $\theta^{-1}(X^{\text{reg}})$ to $\mathcal{Y}^+$ to get a $\mathbb{Q}$-factorial variety with a morphism

$$X_0^+ := \theta^{-1}(X^{\text{reg}}) \cup \mathcal{Y}^+ \longrightarrow X^{\text{reg}} \cup \pi_1(\mathcal{Y}|_U) \ = X \setminus \{q_1, \dots, q_r\}$$

that factors as the composition of a purely inseparable cover and a birational morphism. The finitely many points $q_1, \dots, q_r$ are given by the intersection of $Y_1 \cup \dots \cup Y_l$ with the singular locus of X, since by generality the base locus of $|\Lambda|$ is contained in the regular locus of X.

Let X^+ be a projective closure of X_0^+. Over $X \setminus \{q_1, \dots, q_r\}$ the morphism $\psi\colon X^+ \to X$ agrees with $X_0^+ \to X$, so properties (1) and (2) hold. By construction, the pullback of any member $\neq Y_1, \dots, Y_l$ of $|\Lambda|$ to X^+ is smooth except possibly over $\mathrm{Bs}\,|\Lambda|$. So the pullback of a general member of $|\Lambda|$ is smooth over the singular locus of X, i.e. along the exceptional divisors of ψ, and by openness of smoothness this holds for a general member of $|\mathcal{L}|$ as well. Applying this to sums of b general members of $|\mathcal{L}|$ and again using openness of smoothness, the pullback of a general member of $|\mathcal{L}^b|$ is smooth along the exceptional divisors of ψ. In particular, by Bertini's theorem, the pullback of a general member of $|\mathcal{L}^b|$ is $\mathbb{Q}$-factorial. □

We are now ready to show surjectivity for $\mathcal{L}^2$. Beginning with a divisor on a very general member T of $|\mathcal{L}^2|$, we degenerate to a reducible member. By Lemma 4.3 the divisor on this reducible member will lift to a divisor D^X on X, and we show that up to torsion and $\mathrm{Cl}^0(X)$, the restriction of D^X to T agrees with the original divisor. Even when X is smooth, the family is singular, so to deal with these singularities we apply Lemma 4.5 to ensure that the divisors we work with are $\mathbb{Q}$-Cartier in some necessary places.

Proposition 4.7. *In the setting of Assumption 1, assume that X is defined over a field $\mathbb{K}$ of infinite transcendence degree and that $n \geq 3$. Let S_1, S_2 be complete intersection surfaces in $|V|$ that are very general over k_0, and pick a pencil in $|\mathcal{L}^2|$ through $S_1 + S_2$ that is very general over $\overline{k^{S_1}}$ and $\overline{k^{S_2}}$.*

If T is a complete intersection surface of $n-3$ members of $|V|$ and a member of $|\mathcal{L}^2|$ and if T is very general over the algebraic closure of the compositum of the fields of definition for S_1, S_2, and the pencil, then the restriction map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T)$$

has torsion cokernel.

Proof. We may assume $n = 3$ by Corollary 4.4. Let $S_1, S_2 \in |V|$ be very general over k_0. Then each S_i is a normal surface, $S_1 + S_2$ is a reducible member of $|V| + |V| \subset |\mathcal{L}^2|$, and $C := S_1 \cap S_2$ is a smooth curve contained in the regular locus of X. Let $s \in |\mathcal{L}^2|$ be the point corresponding to $S_1 + S_2$, take a pencil through s that is very general over $\overline{k^{S_1}}$ and $\overline{k^{S_2}}$, and let $\tilde{X} \to \mathbb{P}^1$ be the total space of the pencil. Then $\beta\colon \tilde{X} \to X$ is the blowup along the reducible curve given by the base locus.

The Chow variety $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1)$ parametrizing Weil divisors on the fibers of $\tilde{X} \to \mathbb{P}^1$ [Kol17, 3.21.3] has countably many components not dominating the base. The image of each of these components is a point defined over the algebraic closure of the compositum of k^{S_1}, k^{S_2}, and a field of definition for the pencil. So any $\mathbb{K}$-point g corresponding to a divisor $T = \tilde{X}_g$ very general over this field will not be in the image of any of these components. Let $\{p_j\}$ denote the intersection points $C \cap T$.

Let D_g be an integral divisor on the normal surface T, and consider the component Ω of $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1)$ containing $[D_g]$; by assumption Ω dominates $\mathbb{P}^1$. Let $\Gamma \subset \Omega$ be a curve passing through the point $[D_g]$ and dominating the base $\mathbb{P}^1$. Then the base change $\mathrm{WDiv}(\tilde{X}/\mathbb{P}^1) \times_{\mathbb{P}^1} \Gamma = \mathrm{WDiv}(\tilde{X} \times_{\mathbb{P}^1} \Gamma/\Gamma) \to \Gamma$ admits a section whose image contains the divisor $[D_g]$ on the fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{g}} = \tilde{X}_g$, where $\tilde{g}$ maps to g. That is, we have a divisor $\mathcal{D}$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ whose restriction to the fiber over $\tilde{g}$ is D_g, and whose restriction to any fiber is a 1-cycle on the corresponding complete intersection surface. Denote the base change $\alpha\colon \tilde{X} \times_{\mathbb{P}^1} \Gamma \to \tilde{X}$.

Let $\tilde{s} \in \Gamma(\mathbb{K})$ map to s, and let r be the ramification index of $\Gamma \to \mathbb{P}^1$ at $\tilde{s}$. The fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{s}}$ has two irreducible components $\tilde{S}_1$ and $\tilde{S}_2$, with each $\tilde{S}_i$ mapping isomorphically to S_i. Let $\tilde{p}_j \in \tilde{S}_1 \cap \tilde{S}_2$ denote the point mapping to $p_j \in C$.

The restriction of $\mathcal{D}_{\tilde{s}}$ to each $\tilde{S}_i$ determines a unique Weil divisor class D_i° in each $\mathrm{Cl}(\tilde{S}_i)$. By Lemma 4.8 we know that on $(\tilde{X} \times_{\mathbb{P}^1} \Gamma) \setminus \{\tilde{p}_j\}$ every Weil divisor is locally Cartier along $\tilde{S}_1 \cap \tilde{S}_2 \setminus \{\tilde{p}_j\}$ after multiplication by r. So the difference $D_1^\circ|_{\tilde{S}_1 \cap \tilde{S}_2} - D_2^\circ|_{\tilde{S}_1 \cap \tilde{S}_2}$ is a divisor on $\tilde{S}_1 \cap \tilde{S}_2$ supported on $\{\tilde{p}_j\}$. The local computation also shows that the local class group is $\mathbb{Z} = \langle (\tilde{S}_1)_{\tilde{p}_j} \rangle = \langle (-\tilde{S}_2)_{\tilde{p}_j} \rangle$ at each $\tilde{p}_j$.

We now apply Lemma 4.5 to the very ample line bundle $\mathcal{L}^2|_C$ on C and to the subgroups $A \subset \mathrm{Pic}^0(C)$ and $G \subset \mathrm{Pic}(C)$ generated by the images of the restriction maps

$$\mathrm{Cl}(\tilde{S}_i) \xrightarrow{(\alpha\circ\beta)_*} \mathrm{Cl}(S_i) \longrightarrow \mathrm{Pic}(C)$$

for $i = 0, 1$. By Lemma 2.15 and the Theorem of the Base 2.2, these A and G satisfy the hypotheses of Lemma 4.5, so we conclude that $(\beta_*\alpha_*D_1^\circ)|_C - (\beta_*\alpha_*D_2^\circ)|_C = a\sum p_j \in |\mathcal{L}^{2a}|_C|$ for some $a \in \mathbb{Z}$.

We now consider instead the divisor $\mathcal{D} - a\tilde{S}_1$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$. On the fiber over $\tilde{g}$ this agrees with $\mathcal{D}_{\tilde{g}} = D_g$. Over $\tilde{s}$, the fiber $(\mathcal{D} - a\tilde{S}_1)_{\tilde{s}}$ determines a unique Weil divisor class D_i in each $\mathrm{Cl}(\tilde{S}_i)$, and $\mathcal{D} - a\tilde{S}_1$ is $\mathbb{Q}$-Cartier along C since it becomes trivial in the local class group at each $\tilde{p}_j$. So the restrictions of $\beta_*\alpha_*D_1$ and $\beta_*\alpha_*D_2$ to C agree in $\mathrm{Pic}(C)$, and the divisors $\beta_*\alpha_*D_1$ on S_1 and $\beta_*\alpha_*D_2$ on S_2 satisfy the hypotheses of Lemma 4.3. So we get a unique Weil divisor class $D^X \in \mathrm{Cl}(X)$ such that $D^X|_{S_i} \sim \beta_*\alpha_*D_i$ for $i = 0, 1$. We now have two divisors $D^X|_T$ and D_g on T.

Claim: $D^X|_T - D_g$ is torsion in $\mathrm{Cl}^{\mathrm{ns}}(T)$.

The claim implies surjectivity of $\mathrm{Cl}(X) \to \mathrm{Cl}(T)$ up to torsion, since $\mathrm{Cl}^0(X) \cong \mathrm{Cl}^0(T)$ by Corollary 3.5. So it remains to show the claim.

Let $(\beta\circ\alpha)^*D^X$ denote the divisor $(D^X \times \Gamma)|_{\tilde{X}\times_{\mathbb{P}^1}\Gamma}$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$. The idea of the proof is that $(\beta\circ\alpha)^*D^X - (\mathcal{D} - a\tilde{S}_1)$ is a divisor on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ whose specialization to the reducible fiber $\tilde{s}$ is trivial. So if it is $\mathbb{Q}$-Cartier, then we can conclude by specialization of the Néron–Severi group that a multiple is algebraically trivial on the very general fiber $\tilde{g}$. In general we can reduce to the case where the divisor is $\mathbb{Q}$-Cartier in the necessary places.

First assume X is smooth. Then the generic fiber $\tilde{X}_\eta$ of $\tilde{X} \to \mathbb{P}^1$ is a smooth surface over the function field of $\mathbb{P}^1$. So the generic fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_\xi$ of the base change $\tilde{X} \times_{\mathbb{P}^1} \Gamma \to \Gamma$ is also smooth over the function field of Γ, and the divisor $(\beta\circ\alpha)^*D^X - (\mathcal{D} - a\tilde{S}_1)$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ is Cartier on the generic fiber. It's also $\mathbb{Q}$-Cartier on the reducible fiber $\tilde{S}_1 + \tilde{S}_2$ over $\tilde{s}$, since D^X is Cartier on X and $\mathcal{D} - a\tilde{S}_1$ was chosen to be $\mathbb{Q}$-Cartier (with Cartier index r) along $\tilde{S}_1 \cap \tilde{S}_2$. So $r((\beta\circ\alpha)^*D^X - (\mathcal{D} - a\tilde{S}_1))_\xi$ is in the kernel of

$$\mathrm{Pic}((\tilde{X} \times_{\mathbb{P}^1} \Gamma)_\xi) \longrightarrow \mathrm{NS}((\tilde{X} \times_{\mathbb{P}^1} \Gamma)_\xi) \longrightarrow \mathrm{NS}((\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\bar{\xi}}) \xrightarrow{\mathrm{sp}_{\tilde{s}}} \mathrm{NS}((\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{s}}) \,.$$

The specialization map on Néron–Severi groups is injective up to torsion because intersection numbers specialize, so some multiple is algebraically trivial on the very general fiber. That is, $lr(D^X|_T - D_g) \in \mathrm{Pic}^0(T)$ for some $l \in \mathbb{Z}_{>0}$.

If X is singular, let $\psi\colon X^+ \to X$ be the alteration defined in Lemma 4.6. We have a divisor $(\beta\circ\alpha)^*D^X - (\mathcal{D} - a\tilde{S}_1)$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$, and we now claim that the pullback $((\beta\circ\alpha)^*D^X - (\mathcal{D} - a\tilde{S}_1)) \times_X X^+$ to $\tilde{X} \times_{\mathbb{P}^1} \Gamma \times_X X^+$ is $\mathbb{Q}$-Cartier near the reducible fiber $(\tilde{S}_1 + \tilde{S}_2) \times_X X^+$. For the D^X part, we have that $(\beta\circ\alpha)^*D^X \times_X X^+$ is $\mathbb{Q}$-Cartier near $(\tilde{S}_1 + \tilde{S}_2) \times_X X^+$ because X^+ is $\mathbb{Q}$-factorial except over $\{q_j\}$, and S_1, S_2 do not contain any of the points q_j by generality. Next, the divisor $\mathcal{D} - a\tilde{S}_1$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ is $\mathbb{Q}$-Cartier along $\tilde{S}_1 \cap \tilde{S}_2$ by construction, and away from $S_1 \cap S_2$ the base change $(S_1+S_2)\times_X X^+ \to S_1+S_2$ becomes $\mathbb{Q}$-factorial by property (4). So $\mathcal{D}-a\tilde{S}_1$ is also $\mathbb{Q}$-Cartier after base change.

$((\beta \circ \alpha)^* D^X - (\mathcal{D} - a\tilde{S}_1)) \times_X X^+$ is also $\mathbb{Q}$-Cartier on the geometric generic fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma \times_X X^+)_{\overline{\xi}}$ of $\tilde{X} \times_{\mathbb{P}^1} \Gamma \times_X X^+ \to \Gamma$. The exceptional divisors of $(\tilde{S}_1+\tilde{S}_2)\times_X X^+ \to \tilde{S}_1+\tilde{S}_2$ are specializations of the exceptional divisors of $(\tilde{X} \times_{\mathbb{P}^1} \Gamma \times_X X^+)_{\overline{\xi}} \to (\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\overline{\xi}}$, so there is some (Cartier) divisor E with exceptional support on the geometric generic fiber such that $r(((\beta\circ\alpha)^* D^X - (\mathcal{D}-a\tilde{S}_1))\times_X X^+)_{\overline{\xi}}+E$ becomes trivial after specializing to the fiber $(\tilde{S}_1+\tilde{S}_2)\times_X X^+$ over $\tilde{s}$. Then $r(((\beta\circ\alpha)^* D^X-(\mathcal{D}-a\tilde{S}_1))\times_X X^+)_{\overline{\xi}}+E$ is numerically trivial, so some multiple $lr(((\beta\circ\alpha)^* D^X-(\mathcal{D}-a\tilde{S}_1))\times_X X^+ + E)|_{\tilde{T}_g\times_X X^+}$ is algebraically trivial on the very general fiber. Pushing forward we get $\beta_*\alpha_*(lr((\beta\circ\alpha)^* D^X - (\mathcal{D}-a\tilde{S}_1))|_{\tilde{T}_g}) = D^X|_T - D_g \in \mathrm{Cl}^0(T)$. □

Lemma 4.8. *In the proof of Proposition 4.7, $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ is locally $\mathbb{Q}$-factorial over $C \setminus \{p_i\}$ with Cartier index equal to the ramification index r of $\Gamma \to \mathbb{P}^1$ at $\tilde{s}$.*

Proof. Note that this computation applies to any collection of Cartier divisors $S_1 \in |\mathcal{L}_1|, S_2 \in |\mathcal{L}_2|$, and $T \in |\mathcal{L}_1 \otimes \mathcal{L}_2|$ as long as each one is smooth at x.

Let $x \in C = S_1 \cap S_2 \subset X$ be a closed point. Then x is in the regular locus of X, S_1, and S_2, so we may choose local parameters y_1, y_2, y_3 at x such that S_i is locally defined by y_i at x. Let q be a local equation for T at x; then $x \in T$ if and only if q is in the maximal ideal of $k[[y_1,y_2,y_3]]$, i.e. not a unit. The total space of the pencil is locally $\widehat{\mathcal{O}}_{\tilde{X},x} \cong k[[y_1,y_2,y_3,s]]/(y_1y_2 - sq)$ where s is a local parameter for $\mathcal{O}_{\mathbb{P}^1,s}$.

Let t be a local parameter at $\mathcal{O}_{\Gamma,\tilde{s}}$. Then at a point $\tilde{x}$ in $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ mapping to x

$$\widehat{\mathcal{O}}_{\tilde{X}\times_{\mathbb{P}^1}\Gamma,\tilde{x}} \cong k[[y_1,y_2,y_3,t]]/(y_1y_2 - ut^r q)$$

for some unit u. If $x \notin T$, then $\widehat{\mathcal{O}}_{\tilde{X}\times_{\mathbb{P}^1}\Gamma,\tilde{x}} \cong k[[y_1,y_2,y_3,t]]/((u'q)^{-1}y_1y_2 - t^r)$ is an A_{r-1}-singularity, so its class group is $\mathbb{Z}/r\mathbb{Z}$.

If $x \in T$ then q is in the maximal ideal of $k[[y_1,y_2,y_3]]$. Since $T \in |\mathcal{L}^2|$ is very general, x is not in its singular locus and Y not cut out by any of the same equations as S_1+S_2, so we may assume that our coordinates were chosen so that $q = y_3$. Then $\widehat{\mathcal{O}}_{\tilde{X}\times_{\mathbb{P}^1}\Gamma,\tilde{x}} \cong k[[y_1,y_2,y_3,t]]/(y_1y_2 - ut^r y_3) \cong k[[y_1,y_2,y_n,t]]/(y_1y_2 - (uy_3)t^r)$. This ring has trivial Picard group and its class group is $\mathbb{Z} = \langle (y_1 = t = 0)\rangle = \langle -(y_2 = t = 0)\rangle$. □

Over uncountable fields, Noether–Lefschetz for even multiples ≥ 4 immediately follows from Proposition 3.1 and Proposition 4.7: if $n = 3$ then the restriction map $\mathrm{Cl}(X) \to \mathrm{Cl}(T)$ is injective and is surjective up to torsion for surfaces T outside of a countable union of closed subvarieties of $|\mathcal{L}^2|$.

To get surjectivity for odd multiples in dimension 3, we consider a different type of specialization. Instead of degenerating to a reducible member of $|\mathcal{L}| + |\mathcal{L}|$, we consider a reducible member of $|\mathcal{L}_A^2| + |\mathcal{L}_B|$ and apply Proposition 4.7 to lift divisors from the $|\mathcal{L}_A^2|$ component. To conclude with the same specialization argument as before, we must ensure that our divisors are $\mathbb{Q}$-Cartier on the reducible member (possibly after base change by an alteration of X), so we assume $\mathcal{L}_B$ satisfies the conditions of Lemmas 2.10 and 2.15. This assumption is why the $d = 5$ case will be missing from Corollary 4.14 below.

Proposition 4.9. *Let X be a normal threefold over an algebraically closed field $\mathbb{K}$ of infinite transcendence degree, and let $k_0 \subset \mathbb{K}$ be a field of definition for X. Let $\mathcal{L}_A = \mathcal{L}_{A,0}^{d_A}$ where $\mathcal{L}_{A,0}$ is a very ample line bundle and $d_A \geq 2$ is an integer. Let $\mathcal{L}_B$ be a very ample line bundle such that $\mathcal{L}_B = \mathcal{L}_{B,0}^{d_B}$ for $d_B \geq 2$ and $\mathcal{L}_{B,0}$ ample and basepoint-free. Assume that one of the following holds:*

(1) X is smooth,
(2) $\operatorname{char} \mathbb{K} = 0$, or
(3) $\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B \cong \mathcal{M}^d$ for some very ample line bundle $\mathcal{M}$ and integer d.

Let $S_1', S_2' \in |\mathcal{L}_A|$ be very general over k_0, pick a pencil in $|\mathcal{L}_A^2|$ through $S_1' + S_2'$ that is very general over $\overline{k^{S_1'}}$ and $\overline{k^{S_2'}}$, and let $S_{2A} \in |\mathcal{L}_A^2|$ and $S_B \in |\mathcal{L}_B|$ be very general over the algebraic closure of the compositum of the fields of definition for S_1', S_2', and the pencil. Pick a pencil $|\Lambda| \subset |\mathcal{L}_A^2 \otimes \mathcal{L}_B|$ through $S_{2A} + S_B$ very general over the algebraic closure of the compositum of $k^{S_{2A}}$ and k^{S_B}.

If $T \in |\mathcal{L}_A^{\otimes 2} \otimes \mathcal{L}_B|$ is very general over the algebraic closure of the compositum of the above fields, then the restriction map

$$\operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(T)$$

is surjective up to torsion.

Proof. Let $C = S_{2A} \cap S_B$, and let $\tilde{X}$ be the total space of the pencil $|\Lambda|$. Let s and g denote the points on $|\Lambda| = \mathbb{P}^1$ corresponding to $S_{2A} + S_B$ and T, respectively. By our choice of T, g is not in the image of any component of the Chow variety $\operatorname{WDiv}(\tilde{X}/\mathbb{P}^1)$ not dominating the base.

Let D_g be an integral divisor on T. As in the proof of Proposition 4.7, $\operatorname{WDiv}(\tilde{X}/\mathbb{P}^1) \times_{\mathbb{P}^1} \Gamma \to \Gamma$ admits a section after taking some finite base change $\Gamma \to \mathbb{P}^1$. So we have a divisor $\mathcal{D}$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ that restricts to D_g on the fiber over a point $\tilde{g} \in \Gamma$ mapping to g.

Let $\tilde{s} \in \Gamma$ be a point mapping to s. Then $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{s}} \xrightarrow{\cong} \tilde{X}_s \cong S_{2A} + S_B$, and we denote the components of the reducible fiber $(\tilde{X} \times_{\mathbb{P}^1} \Gamma)_{\tilde{s}}$ by $\tilde{S}_{2A}$ and $\tilde{S}_B$. As in the proof of Proposition 4.7, there is an integer a such that $\mathcal{D} + a\tilde{S}_B$ is $\mathbb{Q}$-Cartier along $\tilde{S}_{2A} \cap \tilde{S}_B$, obtained by applying Lemma 4.5 to the very ample line bundle $(\mathcal{L}_A^2 \otimes \mathcal{L}_B)|_C$ on the curve C and to the subgroups

$A \subset \mathrm{Pic}^0(C)$ and $G \subset \mathrm{Pic}(C)$ generated by the images of the restrictions $\mathrm{Cl}(S_{2A}) \to \mathrm{Pic}(C)$ and $\mathrm{Cl}(S_B) \to \mathrm{Pic}(C)$. We use the assumptions on $\mathcal{L}_B$ here to apply Lemma 2.15 and ensure that $A \subsetneq \mathrm{Pic}^0(C)$.

The divisor $\mathcal{D} + a\tilde{S}_B$ on the family defines a unique Weil divisor class $D_{2A} \in \mathrm{Cl}(\tilde{S}_{2A})$. By Proposition 3.1 and Proposition 4.7 the restriction $\mathrm{Cl}(X) \to \mathrm{Cl}(S_{2A})$ is an injection with torsion cokernel, so for some $c \in \mathbb{Z}_{\geq 1}$ there is a unique divisor class D^X on X such that $D^X|_{S_{2A}} \sim c\beta_*\alpha_* D_{2A}$. By Proposition 3.1 its restriction to S_B agrees with the divisor defined by $\mathcal{D} + a\tilde{S}_B$. Let $(\beta \circ \alpha)^* D^X$ denote $(D^X \times \Gamma)|_{\tilde{X} \times_{\mathbb{P}^1} \Gamma}$.

The divisor $(\beta \circ \alpha)^* D^X - c(\mathcal{D} + a\tilde{S}_B)$ on $\tilde{X} \times_{\mathbb{P}^1} \Gamma$ is trivial on the reducible fiber $\tilde{S}_{2A} + \tilde{S}_B$. If X is smooth or if $\mathrm{char}\,\mathbb{K} = 0$, then the same specialization argument in the proof of Proposition 4.7 implies a multiple of this divisor is algebraically trivial when restricted to the fiber over $\tilde{g}$, and using the same argument we conclude that $\mathrm{Cl}(X) \to \mathrm{Cl}(T)$ is surjective up to torsion.

If X is singular and $\mathrm{char}\,\mathbb{K} = p > 0$, construct a purely inseparable alteration $X^+ \to X$ as in Lemma 4.6 using a general pencil in the very ample linear system $|\mathcal{M}|$. Then $((S_{2A} + S_B) \setminus (S_{2A} \cap S_B)) \times_X X^+$ is $\mathbb{Q}$-factorial, and since $(\beta \circ \alpha)^* D^X$ and $\mathcal{D} + a\tilde{S}_B$ are $\mathbb{Q}$-Cartier along $\tilde{S}_{2A} \cap \tilde{S}_B$ by generality and by choice of a, respectively, the argument of Proposition 4.7 goes through. So we again conclude surjectivity of $\mathrm{Cl}(X) \to \mathrm{Cl}(T)$ up to torsion. □

4.2. **Surjectivity over fields $\neq \overline{\mathbb{F}}_p$.** Following a suggestion of Bjorn Poonen, we will now apply a result of André in characteristic 0 and Ambrosi and Christensen in characteristic p to show that if surjectivity holds over fields of infinite transcendence degree, then it also holds for any field $\neq \overline{\mathbb{F}}_p$. The result we will use is for the Picard rank of smooth families, so we will apply it on a resolution of singularities or a suitable alteration. We first recall their result:

Theorem 4.10 ([And96, Théorème 5.2], [Amb18, Corollary 1.7.1.5], [Chr18, Theorem 1.0.1])**.** *Let $k \neq \overline{\mathbb{F}}_p$ be an algebraically closed field, B a k-scheme of finite type, and $\mathcal{T} \to B$ a smooth morphism. Let η be a generic point of B and $\mathcal{T}_{\overline{\eta}}$ the corresponding geometric generic fiber. Then there exists $b \in B(k)$ such that $\rho(\mathcal{T}_{\overline{\eta}}) = \rho(T_b)$.*

Using [MP12, Proposition 3.6], this means that for such b the specialization map $\mathrm{NS}(\mathcal{T}_{\overline{\eta}}) \to \mathrm{NS}(T_b)$ is an isomorphism if $\mathrm{char}\, k = 0$ and an isomorphism up to p-power torsion if $\mathrm{char}\, k = p > 0$. In the case of a family of surfaces, isomorphism up to torsion can be argued directly as follows.

Corollary 4.11. *Let $k \neq \overline{\mathbb{F}}_p$ be an algebraically closed field, B a variety over k with generic point η, and $\mathcal{T} \to B$ a family of normal surfaces. Then there exists $b \in B(k)$ such that the specialization map $\mathrm{Cl}^{\mathrm{ns}}(\mathcal{T}_{\overline{\eta}}) \to \mathrm{Cl}^{\mathrm{ns}}(T_b)$ is an isomorphism up to torsion.*

Proof. After base change by a purely inseparable dominant morphism $B' \to B$, the family $\mathcal{T} \times_B B'$ admits a simultaneous resolution of singularities $\tilde{\mathcal{T}} \to \mathcal{T} \times_B B'$ such that the exceptional divisors on the closed fibers specialize from the geometric generic fiber (obtained by spreading out a resolution of generic fiber of $\mathcal{T} \to B$ after a purely inseparable base change). For a resolution of singularities of a surface $\pi \colon \tilde{T} \to T$, the rank of $\mathrm{Cl}^{\mathrm{ns}}(T)$ is equal to $\rho(\tilde{T}) - \#\{\text{exceptional curves of } \pi\}$, so by construction it suffices to show the statement for the smooth family $\tilde{\mathcal{T}} \to B'$.

If D^i are divisors on $\tilde{\mathcal{T}}_{\overline{\eta}}$ specializing to D^i_b on the fiber $\tilde{T}_b$ over b, then the rank of the intersection matrix $(D^i_b \cdot D^j_b)$ is independent of $b \in B'$ [Ful98, Example 20.3.6], so $\mathrm{NS}(\tilde{\mathcal{T}}_{\overline{\eta}}) \to \mathrm{NS}(\tilde{T}_b)$ has torsion kernel for any b. Thus the specialization map will be surjective up to torsion if and only if $\rho(\tilde{\mathcal{T}}_{\overline{\eta}}) = \rho(\tilde{T}_b)$, and Theorem 4.10 produces such a k-point b. □

Proposition 4.12. *Let X be a normal variety of dimension $n \geq 3$ defined over an algebraically closed field $k \neq \overline{\mathbb{F}}_p$ and $\mathcal{L}$ a very ample line bundle on X.*

(1) If $\mathcal{L} = \mathcal{L}_0^d$ for a very ample line bundle $\mathcal{L}_0$ and integer $d \geq 2$, then there is a complete intersection surface T_b of $|\mathcal{L}^2|$ with $n-3$ members of $|\mathcal{L}|$ defined over k and such that the restriction map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T_b)$$

has torsion cokernel.

(2) If $n = 3$, further assume that $\mathcal{L} = \mathcal{L}_{A,0}^{\otimes 2d_A} \otimes \mathcal{L}_{B,0}^{d_B}$ for very ample line bundles $\mathcal{L}_{A,0}, \mathcal{L}_{B,0}$ and integers $d_A, d_B \geq 2$ and that one of the following holds:
   - *(a) X is smooth,*
   - *(b) $\mathrm{char}\, k = 0$, or*
   - *(c) $\mathcal{L} \cong \mathcal{M}^d$ for some very ample line bundle $\mathcal{M}$ and integer d.*

   Then there is a divisor $T_b \in |\mathcal{L}|$ defined over k and such that the restriction map

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T_b)$$

has torsion cokernel.

Proof. Let $\mathcal{T} \subset X \times B$ denote the corresponding universal family of surfaces and $\mathcal{T}_\eta$ the generic fiber of the projection onto S. Note that the geometric generic fiber $\mathcal{T}_{\overline{\eta}}$ is normal by Bertini's theorem. Let $\mathbb{K}$ be an algebraically closed field of infinite transcendence degree containing k. Let $\mathcal{L}_{\mathbb{K}}$ denote the pullback to $X_{\mathbb{K}} := X \otimes_k \mathbb{K}$, $\mathcal{T}_{\mathbb{K}} \to B_{\mathbb{K}}$ the corresponding universal family, and $(\mathcal{T}_{\mathbb{K}})_{\overline{\eta}}$ the geometric generic fiber, which agrees with the base change of $\mathcal{T}_\eta$ to the algebraic closure of the function field of $B_{\mathbb{K}}$.

By Corollary 4.11 there is a k-point $b \in B$ such that the specialization $\mathrm{Cl}^{\mathrm{ns}}(\mathcal{T}_{\overline{\eta}}) \to \mathrm{Cl}^{\mathrm{ns}}(T_b)$ is an isomorphism up to torsion, where T_b is the corresponding surface. We have the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{Cl}^{\mathrm{ns}}(X_{\mathbb{K}}) & \longrightarrow & \mathrm{Cl}^{\mathrm{ns}}((\mathcal{T}_{\mathbb{K}})_{\overline{\eta}}) & \overset{\text{Lemma 2.4}}{=\!=\!=} & \mathrm{Cl}^{\mathrm{ns}}(\mathcal{T}_{\overline{\eta}}) \\
\Big\| \text{ Lemma 2.4} & & & & \Big\| \begin{array}{c}\text{after } \otimes\mathbb{Q} \\ \text{Corollary 4.11}\end{array} \\
\mathrm{Cl}^{\mathrm{ns}}(X) & & \longrightarrow & & \mathrm{Cl}^{\mathrm{ns}}(T_b)
\end{array}$$

Moreover, $\mathrm{Cl}^{\mathrm{ns}}(X_{\mathbb{K}}) \to \mathrm{Cl}^{\mathrm{ns}}((\mathcal{T}_{\mathbb{K}})_{\overline{\eta}})$ is injective with torsion cokernel by Proposition 3.1, Proposition 3.5, Proposition 4.7 in case (1), and Proposition 4.9 in case (2). Therefore $\mathrm{Cl}^{\mathrm{ns}}(X) \to \mathrm{Cl}^{\mathrm{ns}}(T_b)$ has torsion cokernel, and using Proposition 3.5 again we conclude that $\mathrm{Cl}(X) \to \mathrm{Cl}(T_b)$ has torsion cokernel. □

In higher dimensions, one can apply the above argument to Corollary 4.4 to show:

Corollary 4.13. *Let X be a normal variety of dimension $n \geq 4$ defined over an algebraically closed field $k \neq \overline{\mathbb{F}}_p$, $\mathcal{L}_0$ a very ample line bundle on X, and $d \geq 2$ an integer. Then there is a divisor Y_b in $|\mathcal{L}_0^d|$ defined over k for which the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(Y_b)$$

(1) is surjective if $\mathrm{char}\, k = 0$, *and*
(2) has p-power torsion cokernel if $\mathrm{char}\, k = p > 0$.

The argument is essentially the same as in Proposition 4.12, but in positive characteristic, instead of the resolution of singularities in Corollary 4.11 we take a smooth p-alteration [Tem17] of the generic fiber after a purely inseparable base change. The finite part of the alteration only contributes p-power torsion [Ful98, Example 1.7.4]; this and the torsion in the cokernel of the specialization map [MP12, Proposition 3.6] are the reasons for the torsion in the characteristic p statement.

When $\mathcal{L}_{A,0} = \mathcal{L}_{B,0}$ in case (2) of Proposition 4.12 (and using (1) for the $d = 4$ case) we obtain

Corollary 4.14. *Let X be a normal threefold over an algebraically closed field $k \neq \overline{\mathbb{F}}_p$ and $\mathcal{L}_0$ a very ample line bundle on X. Then for any integer $d \in \{4\} \cup \mathbb{Z}_{\geq 6}$ the restriction map*

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(T)$$

is surjective up to torsion for very general $T \in |\mathcal{L}_0^d|$.

In particular we recover the statement modulo torsion for $X = \mathbb{P}^3_{\mathbb{C}}$ except for the case of degree 5 surfaces.

References

[Amb18] Emiliano Ambrosi. Specialization of Néron-Severi groups in positive characteristic. *arXiv e-prints*, page arXiv:1810.06481, October 2018.

[And96] Yves André. Pour une théorie inconditionnelle des motifs. *Inst. Hautes Études Sci. Publ. Math.*, 83:5–49, 1996.

[BG12] Ugo Bruzzo and Antonella Grassi. Picard group of hypersurfaces in toric 3-folds. *Internat. J. Math.*, 23(2):1250028, 14, 2012.

[BG18] Ugo Bruzzo and Antonella Grassi. The Noether–Lefschetz locus of surfaces in toric threefolds. *Commun. Contemp. Math.*, 20(5):1750070, 20, 2018.

[BGL20] Ugo Bruzzo, Antonella Grassi, and Angelo Felice Lopez. Existence and Density of General Components of the Noether–Lefschetz Locus on Normal Threefolds. *International Mathematics Research Notices*, 02 2020.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[BN16] John Brevik and Scott Nollet. Grothendieck-Lefschetz theorem with base locus. *Israel J. Math.*, 212(1):107–122, 2016.

[BN20] John Brevik and Scott Nollet. Moishezon's theorem and degeneration. *J. Algebra*, 544:463–482, 2020.

[Cas94] Guido Castelnuovo. Sulle superfici algebriche che ammettono un sistema doppiamente infinito di sezioni piane irriducibili. *Rendic. Acad. Lincei*, 3:22–25, 1894.

[CGGH83] James Carlson, Mark Green, Phillip Griffiths, and Joe Harris. Infinitesimal variations of Hodge structure. I. *Compositio Math.*, 50(2-3):109–205, 1983.

[Cha13] François Charles. The Tate conjecture for $K3$ surfaces over finite fields. *Invent. Math.*, 194(1):119–145, 2013.

[Cho52] Wei-Liang Chow. On Picard varieties. *Amer. J. Math.*, 74:895–909, 1952.

[Cho55] Wei-Liang Chow. Abelian varieties over function fields. *Trans. Amer. Math. Soc.*, 78:253–275, 1955.

[Chr18] Atticus Christensen. Specialization of Néron-Severi groups in characteristic p. *arXiv e-prints*, page arXiv:1810.06550, October 2018.

[Con06] Brian Conrad. Chow's K/k-image and K/k-trace, and the Lang–Néron theorem. *Enseign. Math*, 2006.

[DK73] P. Deligne and N. Katz. *Groupes de monodromie en géométrie algébrique. II.* Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, Berlin-New York, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz.

[Ein85] Lawrence Ein. An analogue of Max Noether's theorem. *Duke Math. J.*, 52(3):689–706, 1985.

[FJ08] Michael D. Fried and Moshe Jarden. *Field arithmetic*, volume 11 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2008. Revised by Jarden.

[FOV99] H. Flenner, L. O'Carroll, and W. Vogel. *Joins and intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[GK19] Mainak Ghosh and Amalendu Krishna. Bertini theorems revisited. *arXiv e-prints*, page arXiv:1912.09076, December 2019.

[Gre84] Mark L. Green. Koszul cohomology and the geometry of projective varieties. *J. Differential Geom.*, 19(1):125–171, 1984.

[Gro05] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, volume 4 of *Documents Mathématiques (Paris) [Mathematical Documents (Paris)]*. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[GS13] Tom Graber and Jason Michael Starr. Restriction of sections for families of abelian varieties. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.*, pages 311–327. Amer. Math. Soc., Providence, RI, 2013.

[Har70] Robin Hartshorne. *Ample subvarieties of algebraic varieties.* Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin-New York, 1970. Notes written in collaboration with C. Musili.

[Igu52] Jun-ichi Igusa. On the Picard varieties attached to algebraic varieties. *Amer. J. Math.*, 74:1–22, 1952.

[Jos95] Kirti Joshi. A Noether-Lefschetz theorem and applications. *J. Algebraic Geom.*, 4(1):105–135, 1995.

[Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[Kol17] János Kollár. Families of varieties of general type. `https://web.math.princeton.edu/~kollar/book/modbook20170720.pdf`, 2017.

[Kol18] János Kollár. Mumford divisors. *arXiv e-prints*, page arXiv:1803.07596, March 2018.

[Kol20] János Kollár. What determines a variety? *arXiv e-prints*, page arXiv:2002.12424, February 2020.

[Lef21] Solomon Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties. *Trans. Amer. Math. Soc.*, 22(3):327–406, 1921.

[LN59] S. Lang and A. Néron. Rational points of abelian varieties over function fields. *Amer. J. Math.*, 81:95–118, 1959.

[Mat52] Teruhisa Matsusaka. On the algebraic construction of the Picard variety. *Proc. Japan Acad.*, 28:5–8, 1952.

[Moi67] B. G. Moishezon. Algebraic homology classes on algebraic varieties. *Izv. Akad. Nauk SSSR Ser. Mat.*, 31:225–268, 1967.

[MP12] Davesh Maulik and Bjorn Poonen. Néron-Severi groups under specialization. *Duke Math. J.*, 161(11):2167–2206, 2012.

[Nér52] André Néron. Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps. *Bull. Soc. Math. France*, 80:101–166, 1952.

[Noe82] M. Noether. Zur Grundlegung der Theorie der algebraischen Raumcurven. *J. Reine Angew. Math.*, 93:271–318, 1882.

[NS52] André Néron and Pierre Samuel. La variété de Picard d'une variété normale. *Ann. Inst. Fourier (Grenoble)*, 4:1–30 (1954), 1952.

[RS06] G. V. Ravindra and V. Srinivas. The Grothendieck-Lefschetz theorem for normal projective varieties. *J. Algebraic Geom.*, 15(3):563–590, 2006.

[RS09] G. V. Ravindra and V. Srinivas. The Noether-Lefschetz theorem for the divisor class group. *J. Algebra*, 322(9):3373–3391, 2009.

[Ser92] Jean-Pierre Serre. *Topics in Galois theory*, volume 1 of *Research Notes in Mathematics*. Jones and Bartlett Publishers, Boston, MA, 1992. Lecture notes prepared by Henri Damon [Henri Darmon], With a foreword by Darmon and the author.

[Shi83] Tetsuji Shioda. Algebraic cycles on a certain hypersurface. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 271–294. Springer, Berlin, 1983.

[Tem17] Michael Temkin. Tame distillation and desingularization by p-alterations. *Ann. of Math. (2)*, 186(1):97–126, 2017.

[Ter85] Tomohide Terasoma. Complete intersections with middle Picard number 1 defined over $\mathbf{Q}$. *Math. Z.*, 189(2):289–296, 1985.

[Voi88] Claire Voisin. Une précision concernant le théorème de Noether. *Math. Ann.*, 280(4):605–611, 1988.

[Voi89] Claire Voisin. Composantes de petite codimension du lieu de Noether-Lefschetz. *Comment. Math. Helv.*, 64(4):515–526, 1989.

[Voi03] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.

[Voi20] Claire Voisin. Letter to János Kollár, 2020.

[Wei46] André Weil. *Foundations of Algebraic Geometry*. American Mathematical Society Colloquium Publications, vol. 29. American Mathematical Society, New York, 1946.

[Wei50] André Weil. Variétés abéliennes. In *Algèbre et théorie des nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24*, pages 125–127. Centre National de la Recherche Scientifique, Paris, 1950.

[Wei54] André Weil. Sur les critères d'équivalence en géométrie algébrique. *Math. Ann.*, 128:95–127, 1954.

1855 East Hall, 530 Church Street, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

E-mail address: lenaji@umich.edu