Geometrically non-reduced varieties

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General fibers of morphisms

Let k be an algebraically closed field.

In characteristic 0, fibers of morphisms behave well:

Theorem (Bertini's theorem)

If $f : \mathcal{X} \to S$ is a morphism of smooth varieties over a field of char 0*, then the general fiber is smooth.*

But in char $p > 0$, can have morphisms $f : \mathcal{X} \to S$ of smooth varieties where all fibers are bad.

Bad fibers in char $p > 0$

Example (Non-reduced fibers)

char $k = p > 0$, $\mathcal{X} = (sx^p + ty^p + 1 = 0) \subset \mathbb{A}^4_{(x,y,s,t)}$, and $f \colon \mathcal{X} \to \mathbb{A}^2_{(s,t)}$.
Fiber over $(a, b) \in \mathbb{A}^2_{(s,t)}$ is
$(ax^p + by^p + 1 = 0) = ((a^{1/p}x + b^{1/p}y + 1)^p = 0) \subset \mathbb{A}^2_{(x,y)}$.

In the above example, fibers are Fano $\iff$ $p = 2$

Example (Reduced, non-normal fibers)

char $k = 3$, $\mathcal{X} = (x^3 + y^2 + t = 0) \subset \mathbb{A}^3_{(x,y,t)}$, and $f \colon \mathcal{X} \to \mathbb{A}^1_t$.
Fiber over $a \in \mathbb{A}^1_t$ is $(x^3 + y^2 + a = 0) = ((x + a^{1/3})^3 + y^2 = 0) \subset \mathbb{A}^2_{(x,y)}$.

Geometric generic fiber

$$\text{General fiber of } \mathcal{X} \to S \longleftrightarrow \text{geometric generic fiber } \mathcal{X}_\eta \otimes_{k(S)} \overline{k(S)}$$

Study the generic fiber $X := \mathcal{X}_\eta$ of $\mathcal{X} \to S$
X is a variety over the imperfect field $K = k(S)$
(We'll assume X geometrically irreducible, i.e. $X \otimes_K \overline{K}$ irreducible)

- Base changing by a separable field extension preserves reducedness, normality, smoothness
- If char $k = p > 0$ then $\overline{K}/K$ is never separable
- $X \otimes_K K^{1/p^\infty}$ may be non-normal or even non-reduced

Break into pieces: What happens after one Frobenius base change?

$$\begin{array}{ccccccc} Y := (X \otimes_K K^{1/p})^{\nu}_{\text{red}} & \xrightarrow{\text{normalization}} & (X \otimes_K K^{1/p})_{\text{red}} & \hookrightarrow & X \otimes_K K^{1/p} & \longrightarrow & X \\ & & & & \downarrow & & \downarrow \\ & & & & \operatorname{Spec} K^{1/p} & \longrightarrow & \operatorname{Spec} K \end{array}$$

Bad behavior in small p

$$
\begin{array}{ccccccc}
Y := (X \otimes_K K^{1/p})^{\nu}_{\mathrm{red}} & \xrightarrow{\text{normalization}} & (X \otimes_K K^{1/p})_{\mathrm{red}} & \hookrightarrow & X \otimes_K K^{1/p} & \longrightarrow & X \\
 & & & & \downarrow & & \downarrow \\
 & & & & \operatorname{Spec} K^{1/p} & \longrightarrow & \operatorname{Spec} K
\end{array}
$$

Study relationship between ω_X and ω_Y to bound bad behavior of certain types of fibrations (e.g. fibrations of the MMP)? Can you bound it to small primes?

Example (Curve case)

$$
p_a(X) = 0 \implies \begin{cases} X \cong \mathbb{P}^1_K, \text{ or} \\ p = 2 \text{ and } X \otimes_K \overline{K} = \text{double line} \end{cases}
$$

Tate's genus change formula

$$Y := (X \otimes_K K^{1/p})^{\nu}_{\mathrm{red}} \xrightarrow{\text{normalization}} (X \otimes_K K^{1/p})_{\mathrm{red}} \hookrightarrow X \otimes_K K^{1/p} \longrightarrow X$$

Theorem (Tate '52)

If X is a regular geometrically reduced curve over K and $Y = (X \otimes_K K^{1/p})^{\nu}$, then $\deg(\omega_X) - \deg(\omega_Y)$ is a positive integer divisible by $p-1$.

Corollary

$$p_a(X) = 1 \implies \begin{cases} X \text{ smooth elliptic curve, or} \\ p = 2 \text{ or } 3 \text{ and } Y \cong \mathbb{P}^1_{K^{1/p}} \end{cases}$$

Proof.

$p-1$ divides $\deg(\omega_X) - \deg(\omega_Y) = -\deg \omega_Y = 2 - 2p_a(Y) > 0$
$\implies p_a(Y) = 0$ and $p = 2$ or 3. □

Previous results

- Schröer '08: scheme-theoretic proof of Tate's genus change formula

Notation

X normal variety over a field K
$\phi\colon Y \to X$ normalization of $(X \otimes_K K^{1/p})_{\text{red}}$

- Tanaka '18:
  If X is regular, then $\phi^*\omega_X - \omega_Y$ is effective.
  $\phi^*\omega_X - \omega_Y$ trivial $\iff$ $X \otimes_K K^{1/p}$ is normal.
  $X \otimes_K K^{1/p}$ reduced $\implies$ $\mathrm{Supp}(\phi^*\omega_X - \omega_Y)$ is the conductor.
- Patakfalvi–Waldron '20:
  $\phi^*\omega_X - \omega_Y \sim (p-1)C$ for some effective Weil divisor C on Y
  If $X \otimes_K K^{1/p}$ is reduced, then C can be chosen so that $(p-1)C$ is the divisorial part of the conductor of the normalization.

Question

Can we say more about C when X is geometrically non-reduced?

A canonical bundle formula

Theorem (— –Waldron)

Let K be the function field of a variety over $k = \overline{k}$ of characteristic $p > 0$.
Let X be a normal, geometrically integral variety over K, and let $\phi\colon Y \to X$ be the normalization of $(X \otimes_K K^{1/p})_{\mathrm{red}}$.
Then there is a canonically determined linear system $\mathfrak{C}$ of Weil divisors with fixed part $\mathfrak{F}$ and movable part $\mathfrak{M}$ such that

$$\phi^*\omega_X - \omega_Y \sim (p-1)\mathfrak{C}.$$

- $\mathfrak{M}$ *"measures the non-reducedness" of $X \otimes_K K^{1/p}$*
- $\mathfrak{F}$ *"measures the non-normality" of $(X \otimes_K K^{1/p})_{\mathrm{red}}$*
- $\mathfrak{M}$ *induces a rational map $f : X \dashrightarrow V$ that satisfies a certain universal property for reducedness*

Geometric consequence for Fanos

Corollary

If X is the generic fiber of a Mori fiber space and $p - 1 > 2 \dim X$, then any geometric non-reducedness of X "comes from a lower dimension."

Inductive strategy for showing geometric non-reducedness for $p \gg 0$:

1. Apply Corollary to get $X \overset{\text{rel dim}\geq 1}{\dashrightarrow} V$ where the non-reducedness comes from
2. Show V is "Fano-ish"
3. Run MMP on V, apply Corollary again

Steps 2 and 3 have issues in char $p > 0$

Proof idea of Fano corollary

Conclusion of Corollary

Up to birational iso, $\exists$ contraction $X \to V$ with $\dim V < \dim X$ such that $(X \otimes_K K^{1/p})_{\mathrm{red}}$ is birational to $X \times_V ((V \otimes_K K^{1/p})_{\mathrm{red}})$

Idea of proof.

Let $f\colon X \dashrightarrow V$ be the map induced by $\mathfrak{M}$ from Main Theorem

- $X \times_V ((V \otimes_K K^{1/p})_{\mathrm{red}})$ is generically reduced (by Main Theorem)

$$\left.\begin{array}{c}\text{Mori's bend and break } + \\ \text{Canonical bundle formula with } p-1\end{array}\right\} \Longrightarrow \begin{array}{c} f \text{ not generically finite} \\ \text{if } p-1 > 2\dim X\end{array}$$

$$\begin{array}{ccccc}
(X \otimes_K K^{1/p})_{\mathrm{red}} & \overset{\simeq}{\dashrightarrow} & X \times_V ((V \otimes_K K^{1/p})_{\mathrm{red}}) & \longrightarrow & X \\
 & & \downarrow & & \downarrow {\scriptstyle f \,|\, \text{rel dim} \geq 1} \\
 & & (V \otimes_K K^{1/p})_{\mathrm{red}} & \longrightarrow & V
\end{array}$$

□

Rough idea of main theorem: Quotients by foliations

$$\phi^*\omega_X - \omega_Y \sim (p-1)(\mathfrak{F} + \mathfrak{M})$$

A foliation on a variety Y is a p-closed saturated subsheaf $\mathcal{F} \subset \mathcal{T}_Y$

$$\{\text{foliations } \mathcal{F} \subset \mathcal{T}_Y\} \overset{\text{1-to-1}}{\longleftrightarrow} \left\{ \begin{matrix} \text{height one purely inseparable} \\ \text{morphisms to normal varieties } Y \to X \end{matrix} \right\}$$

Canonical bundle formula for foliations (Ekedahl '85): $\omega_{Y/X} \cong \det \mathcal{F}^{\otimes(p-1)}$

- Patakfalvi–Waldron '20: $-\det \mathcal{F} \geq 0$

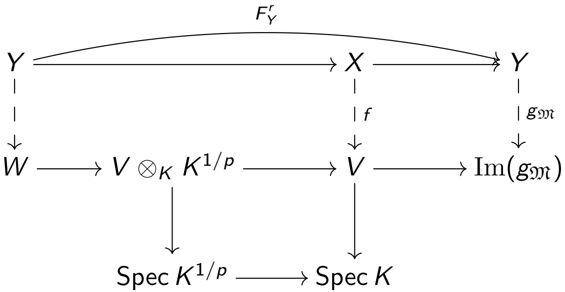
$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & (X \otimes_K K^{1/p})_{\text{red}} & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} K^{1/p} & \longrightarrow & \operatorname{Spec} K \end{array}$$

We define

- $\mathfrak{M}$ coming from $(X \otimes_K K^{1/p})_{\text{red}} \to X$
- $\mathfrak{F}$ coming from the normalization $Y \to (X \otimes_K K^{1/p})_{\text{red}}$

Bulk of the work is verifying that $\mathfrak{M}$ and $\mathfrak{F}$ satisfy our list of properties

Some properties of the movable part $\mathfrak{M}$



$f : X \dashrightarrow V$ = Stein factorization of $X \dashrightarrow \mathrm{Im}(g_{\mathfrak{M}})$
W = normalization of $(V \otimes_K K^{1/p})_{\mathrm{red}}$

- $X_\eta \otimes_{k(V)} k(W)$ is reduced (where X_η = generic fiber of f)
- $\mathfrak{M}$ is the pullback of $\mathfrak{M}_W$
- If $X \dashrightarrow Z$ is a rational map whose generic fiber stays reduced after base changing to $k((Z \otimes_K K^{1/p})_{\mathrm{red}})$, then $X \dashrightarrow Z \dashrightarrow V$

Thank you!

Thank you for your attention!