Geometrically non-reduced varieties

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General fibers of morphisms

Notation

k will denote an algebraically closed field throughout this talk

In characteristic 0, fibers of morphisms behave well:

Theorem (Bertini's theorem)

If $\mathcal{X} \to S$ is a morphism of smooth varieties over a field of char 0, *then the general fiber is smooth.*

But in char $p > 0$, there exist morphisms $\mathcal{X} \to S$ of smooth varieties where all fibers are bad!

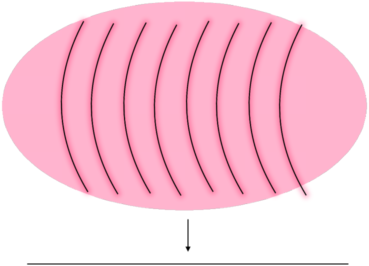
Bad fibers in char $p > 0$

Example (Non-reduced fibers)

- char $k = p > 0$ and $\mathcal{X} = (sx^p + ty^p + 1 = 0) \subset \mathbb{A}^4_{(x,y,s,t)}$
- $\mathcal{X} \to \mathbb{A}^2_{(s,t)}$

Fiber over $(a, b) \in \mathbb{A}^2_{(s,t)}$ is $ax^p + by^p + 1 = (a^{1/p}x + b^{1/p}y + 1)^p = 0$
$\implies$ every fiber is a non-reduced one-dimensional subscheme of $\mathbb{A}^2_{(x,y)}$

Note: Fibers are Fano $\iff$ $p = 2$

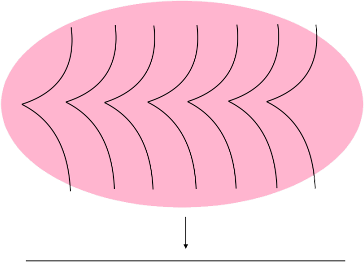


Bad fibers in char $p > 0$

Example (Reduced, non-normal fibers)

- char $k = 3$ and $\mathcal{X} = (x^2 + y^3 - t = 0) \subset \mathbb{A}^3_{(x,y,t)}$
- $\mathcal{X} \to \mathbb{A}^1_t$

Fiber over $a \in \mathbb{A}^1_t$ is $x^2 + y^3 - a = x^2 + (y - a^{1/3})^3 = 0$
$\implies$ every fiber is a singular, cuspidal curve in $\subset \mathbb{A}^2_{(x,y)}$



Geometric generic fiber

$$\text{General fiber of } \mathcal{X} \to S \longleftrightarrow \text{geometric generic fiber } \mathcal{X}_\eta \otimes_{k(S)} \overline{k(S)}$$
$$\text{Properties of general fibers} \longleftrightarrow \text{geometric properties of generic fiber}$$

Study the generic fiber $X := \mathcal{X}_\eta$ of $\mathcal{X} \to S$

Notation

$X = \mathcal{X}_\eta$ is a variety over the function field $L = k(S)$

(We'll assume X geometrically irreducible, i.e. $X \otimes_L \overline{L}$ irreducible)

- Separable base change preserves reducedness, normality, regularity
- If char $k = p > 0$ then $\overline{L}/L$ is never separable, i.e. L is an imperfect field
- $X \otimes_L L^{1/p^\infty}$ may be non-normal or even non-reduced
- All the problems happen after some finite base change $\otimes_L L^{1/p^m}$

Break into pieces: What happens after one Frobenius base change?

$$\begin{tikzcd}
Y := (X \otimes_L L^{1/p})^\nu_{\mathrm{red}} \arrow[r, "\text{normalization}"] & (X \otimes_L L^{1/p})_{\mathrm{red}} \arrow[r, hook] & X \otimes_L L^{1/p} \arrow[r] \arrow[d] & X \arrow[d] \\
& & \operatorname{Spec} L^{1/p} \arrow[r] & \operatorname{Spec} L
\end{tikzcd}$$

Bad behavior in small p

Notation

From now on:

- $k = \overline{k}$ has char $k = p > 0$
- $L =$ function field of a variety over k

$$\begin{tikzcd}
Y := (X \otimes_L L^{1/p})^{\nu}_{\mathrm{red}} \arrow[r, "\text{normalization}"] & (X \otimes_L L^{1/p})_{\mathrm{red}} \arrow[r, hook] & X \otimes_L L^{1/p} \arrow[r] \arrow[d] & X \arrow[d] \\
& & \operatorname{Spec} L^{1/p} \arrow[r] & \operatorname{Spec} L
\end{tikzcd}$$

Study relationship between K_X and K_Y to bound bad behavior of certain types of fibrations (e.g. fibrations of the MMP)? Can you bound it to small primes?

Example (Curve case)

$$p_a(X) = 0 \implies \begin{cases} X \otimes_L \overline{L} \cong \mathbb{P}^1_{\overline{L}}, \text{ or} \\ p = 2 \text{ and } X \otimes_L \overline{L} = \text{double line} \end{cases}$$

$$Y := (X \otimes_L L^{1/p})^{\nu}_{\mathrm{red}} \xrightarrow{\text{normalization}} (X \otimes_L L^{1/p})_{\mathrm{red}} \hookrightarrow X \otimes_L L^{1/p} \longrightarrow X$$

Theorem (Tate '52)

If X is a regular geometrically reduced curve over L and $Y = (X \otimes_L L^{1/p})^{\nu}$, then $\deg(K_X) - \deg(K_Y)$ is a positive integer divisible by $p - 1$.

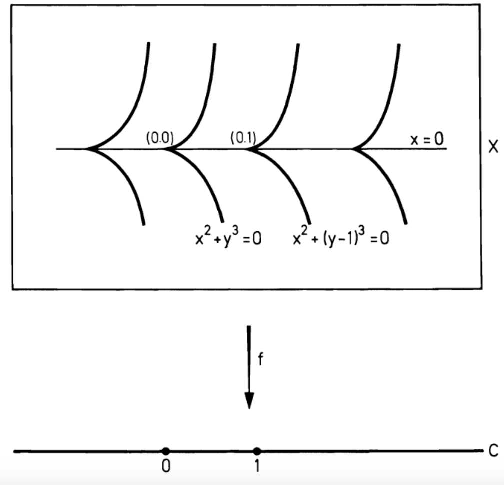
Corollary

$$p_a(X) = 1 \implies \begin{cases} X \text{ smooth elliptic curve, or} \\ p = 2 \text{ or } 3 \text{ and } Y \cong \mathbb{P}^1_{L^{1/p}} \end{cases}$$

Proof.

$p - 1$ divides $\deg(K_X) - \deg(K_Y) = -\deg K_Y = 2 - 2p_a(Y) > 0$
$\implies p_a(Y) = 0$ and $p = 2$ or 3. $\square$

Quasi-elliptic surfaces



Bombieri–Mumford, Enriques' Classification of Surfaces in Char. p, III. (1976)

This proves Proposition 1, plus the old result of Tate [13] that such a line of cusps is only possible in characteristic 2 or 3. Q.E.D.

Tate's genus change formula

GENUS CHANGE IN INSEPARABLE EXTENSIONS OF FUNCTION FIELDS

JOHN TATE

COROLLARY 1. *If k is a field of algebraic functions of one variable of characteristic $p>0$ and genus g, and K is a totally inseparable finite extension of k of genus G, then $G-g$ is divisible by $(p-1)/2$.*

REMARK. A simple example of the situation we are discussing is the case where $k=k_0(x, y)$ is a hyperelliptic field generated by an equation of the form $y^2=x^p-a$ $(p\neq 2)$, of genus $(p-1)/2$. Upon adjunction of $a^{1/p}$ we obtain a rational field of genus 0. Corollary 1 shows that this genus drop is typical.

- Schröer '08: scheme-theoretic proof of Tate's genus change formula

Previous results

Notation

X normal variety over a field $L = H^0(X, \mathcal{O}_X)$
$\phi\colon Y \to X$ normalization of $(X \otimes_L L^{1/p})_{\mathrm{red}}$

- Tanaka '18: (Assuming X regular)
  - $\phi^* K_X - K_Y$ is effective
  - $\phi^* K_X - K_Y$ trivial $\iff$ $X \otimes_L L^{1/p}$ is normal
  - $X \otimes_L L^{1/p}$ reduced $\implies$ $\mathrm{Supp}(\phi^* K_X - K_Y)$ is the conductor
- Patakfalvi–Waldron '20:
  - $\phi^* K_X - K_Y \sim (p-1)C$ for some effective Weil divisor C on Y
  - $X \otimes_L L^{1/p}$ reduced $\implies$ can choose C so that $(p-1)C$ is the divisorial part of the conductor

Question

Can we say more about C when X is geometrically non-reduced?

A canonical bundle formula

Theorem (— –Waldron)

Let L be the function field of a variety over $k = \overline{k}$ of characteristic $p > 0$. Let X be a normal variety over a field $L = H^0(X, \mathcal{O}_X)$, let $L \subset L' \subset L^{1/p}$ be a field extension, and let $\phi\colon Y \to X$ be the normalization of $(X \otimes_L L')_{\mathrm{red}}$. Then there is a canonically determined linear system $\mathfrak{C}_{L'/L}$ of Weil divisors with fixed part $\mathfrak{F}_{L'/L}$ and movable part $\mathfrak{M}_{L'/L}$ such that

$$\phi^* K_X - K_Y \sim (p-1)\mathfrak{C}_{L'/L}$$

- $\mathfrak{M}_{L'/L}$ *"measures the non-reduced singularities" of $X \otimes_L L'$*
- $\mathfrak{F}_{L'/L}$ *"measures the non-normal singularities" of $(X \otimes_L L')_{\mathrm{red}}$*
- $\mathfrak{M}_{L'/L}$ *induces a rational map $f : X \dashrightarrow V$ that satisfies a certain universal property for reducedness for base change by L'/L*

Non-normal example

- $L = k(s,t)$ and $L' = k(s, t^{1/p})$
- $X = (sx^p + ty^p + z^p = 0) \subset \mathbb{P}^2_L$

Base change diagram for L'/L:

$$\begin{tikzcd}
\begin{array}{c} Y \subset \mathbb{P}^2_{k(s^{1/p},t^{1/p})} \\ s^{1/p}x + t^{1/p}y + z = 0 \end{array} \arrow[r, "\nu"] \arrow[d] & \begin{array}{c} X \otimes_L L' \subset \mathbb{P}^2_{k(s,t^{1/p})} \\ sx^p + (t^{1/p}y + z)^p = 0 \end{array} \arrow[r] \arrow[d] & \begin{array}{c} X \subset \mathbb{P}^2_{k(s,t)} \\ sx^p + ty^p + z^p = 0 \end{array} \arrow[d] \\
L'(s^{1/p}) \arrow[r] & L' \arrow[r] & L
\end{tikzcd}$$

- $X \otimes_L L'$ is integral and Y is its normalization
- $\phi^* K_X - K_Y \sim \mathcal{O}_Y(p-1) \implies C \sim \mathcal{O}_Y(1)$
- $\mathfrak{F}_{L'/L} = (x = 0) \in |\mathcal{O}_Y(1)|$
- $\mathfrak{M}_{L'/L} = 0$

Non-reduced example

- $L = k(s_0, \dots, s_n)$ and $L' = L^{1/p}$
- $X = \left((\sum_{i=0}^{n} s_i x_i^p) + x_{n+1}^p = 0 \right) \subset \mathbb{P}_L^{n+1}$

Base change diagram for $L^{1/p}/L$:

$$
\begin{tikzcd}
Y \subset \mathbb{P}^{n+1}_{k(s_0^{1/p},\dots,s_n^{1/p})} \arrow[r, "\mathrm{red}"] & X \otimes_L L^{1/p} \subset \mathbb{P}^{n+1}_{k(s_0^{1/p},\dots,s_n^{1/p})} \arrow[r] & X \subset \mathbb{P}^{n+1}_{k(s_0,\dots,s_n)} \\
\sum_{i=0}^{n} s_i^{1/p} x_i + x_{n+1} = 0 \arrow[dr] & (\sum_{i=0}^{n} s_i^{1/p} x_i + x_{n+1})^p = 0 \arrow[d] & \sum_{i=0}^{n} s_i x_i^p + x_{n+1}^p \arrow[d] \\
& L^{1/p} & L \arrow[l]
\end{tikzcd}
$$

- $X \otimes_L L^{1/p}$ is non-reduced and $(X \otimes_L L^{1/p})_{\mathrm{red}}$ is normal
- $\phi^* K_X - K_Y \sim \mathcal{O}_Y(p-1) \implies C \sim \mathcal{O}_Y(1)$
- $\mathfrak{F}_{L^{1/p}/L} = 0$
- $\mathfrak{M}_{L^{1/p}/L} = \langle x_i \mid 0 \le i \le n \rangle \subset |\mathcal{O}_Y(1)|$

Non-reduced example, continued

- $L = k(s_0, \dots, s_n)$ and $L' = L^{1/p}$
- $X = \left((\sum_{i=0}^{n} s_i x_i^p) + x_{n+1}^p = 0\right) \subset \mathbb{P}_L^{n+1}$
- $Y = \left((\sum_{i=0}^{n} s_i^{1/p} x_i) + x_{n+1} = 0\right) \subset \mathbb{P}_{L^{1/p}}^{n+1}$
- $\mathfrak{M}_{L^{1/p}/L} = \langle x_i \mid 1 \le i \le n \rangle \subset |\mathcal{O}_Y(1)|$

$$\begin{tikzcd}
Y = \operatorname{Proj} \frac{L^{1/p}[x_0,\dots,x_{n+1}]}{(\sum_{i=0}^{n} s_i^{1/p} x_i + x_{n+1})} \arrow[r, "\phi"'] \arrow[d, "|\mathfrak{M}_{L^{1/p}/L}|"] \arrow[rr, bend left, "F_Y"] & X \arrow[r] \arrow[dr] & Y^p \arrow[d] \\
\operatorname{Proj} L^{1/p}[x_0, \dots, x_n] & & \operatorname{Proj} L[x_0^p, \dots, x_n^p]
\end{tikzcd}.$$

- Stein factorization $X \xrightarrow{f} V \to \operatorname{Proj} L[x_0^p, \dots, x_n^p]$ is $X = V$
- Factorization from the main theorem $X \xrightarrow{f} V$ induced by $\mathfrak{M}_{L^{1/p}/L}$ is trivial
- Note: X is Fano if $p < n + 2$

Geometric consequence for Fanos

Corollary

If X is the generic fiber of a Mori fiber space and $p-1>2\dim X$, then any geometric non-reducedness of X "comes from a lower dimension."

Inductive strategy for showing geometric reducedness for $p \gg 0$:

1. Apply Corollary to get $X \overset{\text{rel dim}\geq 1}{\dashrightarrow} V$ where the non-reducedness comes from
2. Show V is "Fano-ish"
3. Run MMP on V, apply Corollary again

Steps 2 and 3 have issues in char $p>0$
(See Tanaka's "Pathologies" paper)

Proof idea of Fano corollary

Conclusion of Corollary ($p-1>2\dim X$)

Up to birational iso, $\exists$ contraction $X\to V$ with $\dim V<\dim X$ such that $(X\otimes_L L^{1/p})_{\mathrm{red}}$ is birational to $X\times_V((V\otimes_L L^{1/p})_{\mathrm{red}})$

False for small p: $\exists$ Fano X with $X=V$ for any $p<\dim X+2$

Idea of proof.

Let $f\colon X\dashrightarrow V$ be the map induced by $\mathfrak{M}_{L^{1/p}/L}$ from Main Theorem

- $X\times_V((V\otimes_L L^{1/p})_{\mathrm{red}})$ is generically reduced (by Main Theorem)

$$\left.\begin{array}{c}\text{Mori's bend and break }+\\ \text{Canonical bundle formula with } p-1\end{array}\right\}\implies \begin{array}{c}f\text{ not generically finite}\\ \text{if } p-1>2\dim X\end{array}$$

$$\begin{tikzcd}
(X\otimes_L L^{1/p})_{\mathrm{red}} \arrow[r, dashed, "\simeq"] & X\times_V((V\otimes_L L^{1/p})_{\mathrm{red}}) \arrow[r] \arrow[d, dashed] & X \arrow[d, dashed, "f"', "\text{rel dim}\geq 1"] \\
& (V\otimes_L L^{1/p})_{\mathrm{red}} \arrow[r] & V
\end{tikzcd}$$

Tool for main theorem: Quotients by foliations

Main Theorem

$$\phi^* K_X - K_Y \sim (p-1)(\mathfrak{F} + \mathfrak{M})$$

Definition

A foliation on a variety Y is a p-closed saturated subsheaf $\mathcal{F} \subset \mathcal{T}_{Y/k}$

- $\mathcal{T}_{Y/k}$ identifies with the sheaf of k-derivations on Y
- k-derivation on Y is locally a k-linear map $\Delta\colon \mathcal{O}_Y \to \mathcal{O}_Y$ satisfying the Leibniz rule $\Delta(xy) = x\Delta(y) + \Delta(x)y$
- In char p, composing Δ with itself p times also satisfies the Leibniz rule $\implies \Delta^p$ is a derivation
- p-closed means closed under p^{th} powers

Observe: $\Delta(x^p) = px^{p-1}\Delta(x) = 0$ for all $x \in \mathcal{O}_Y$
$\implies$ any derivation on Y kills $\mathcal{O}_Y^p$

Foliations

Theorem (Jacobson '44, Rudakov–Shafarevich '76, Ekedahl '83)

Let Y be a normal variety over k. There is a 1-to-1 correspondence

$$\{\textit{foliations } \mathcal{F} \subset \mathcal{T}_Y\} \xleftrightarrow{\text{1-to-1}} \left\{ \begin{array}{c} \textit{height one purely inseparable morphisms} \\ \textit{to normal varieties } Y \to X \to Y^p \end{array} \right\}$$

- Given $\mathcal{F}$, define $Y \xrightarrow{\phi} X = \underline{\mathrm{Spec}}_{Y^p}(\text{subsheaf of } \mathcal{O}_Y \text{ killed by } \mathcal{F})$
- Given $Y \to X$, then $\mathcal{T}_{Y/X} \subset \mathcal{T}_{Y/k}$ is a foliation

Example: Frobenius $Y \xrightarrow{F_{Y/k}} Y^p$ corresponds to the foliation $\mathcal{T}_{Y/k}$

Canonical bundle formula for foliations (Ekedahl '85)

If $Y \xrightarrow{/\mathcal{F}} X$ is a height one purely inseparable morphism of normal varieties, then

$$K_{Y/X} \cong \det \mathcal{F}^{\otimes[p-1]}$$

Patakfalvi–Waldron: $\phi^* K_X - K_Y \sim (p-1)C$

- X normal variety over L such that $H^0(X, \mathcal{O}_X) = L$
- $L \subset L' \subset L^{1/p} \implies \phi\colon Y \to X$ is height one (i.e. F_Y factors $Y \xrightarrow{\phi} X \to Y^p$)

$$\begin{tikzcd}
Y \arrow[r, "\nu \circ \mathrm{red}"'] \arrow[rr, bend left, "\phi = /\mathcal{F}"] \arrow[dr, "g"'] & X \otimes_L L' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \operatorname{Spec} L' \arrow[r] & \operatorname{Spec} L
\end{tikzcd}$$

$\mathcal{F} = \mathcal{T}_{Y/X} \subset \mathcal{T}_{Y/k}$ foliation corresponding to ϕ
Patakfalvi–Waldron '20 show $\phi^* K_X - K_Y \sim (p-1)C$ with $C \geq 0$:

- They show composition $\mathcal{F} \hookrightarrow \mathcal{T}_{Y/k} \to g^* \mathcal{T}_{L'/k}$ is injective
  (Second map from cotangent sequence of $Y \to L' \to k$)
- Moreover

$$\mathcal{F} \hookrightarrow g^* \mathcal{T}_{L'/L} \cong \mathcal{O}_Y^{\oplus s}$$

$$\implies \exists \text{ nonzero } C \in H^0(Y, -\det \mathcal{F})$$

- Formula comes from relative canonical bundle formula for foliations:

$$-K_{Y/X} \sim (p-1)(-\det \mathcal{F})$$

Our extension: canonically determined linear system $\subset |C|$

Main Theorem

$$\phi^* K_X - K_Y \sim (p-1)(\mathfrak{F} + \mathfrak{M})$$

- Movable part $\mathfrak{M}$ "measures non-reducedness" of $X \otimes_L L'$
- Fixed part $\mathfrak{F}$ "measures non-normality of $(X \otimes_L L')_{\mathrm{red}} =: Z$"

$$\begin{tikzcd}
Y \arrow[r, "\nu"'] \arrow[rrr, bend left, "\phi"] \arrow[drr, "g"'] & Z = (X \otimes_L L')_{\mathrm{red}} \arrow[r, "\mathrm{red}"'] \arrow[dr] & X \otimes_L L' \arrow[r] \arrow[d] & X \arrow[d] \\
 & & \operatorname{Spec} L' \arrow[r] & \operatorname{Spec} L
\end{tikzcd}$$

We define

- $\mathfrak{M}$ coming from $Z \to X$
- $\mathfrak{F}$ coming from the normalization $Y \to Z$

Bulk of work is verifying that $\mathfrak{M}$ and $\mathfrak{F}$ satisfy our list of properties

Properties of $\mathfrak{F}_{L'/L}$ and $\mathfrak{M}_{L'/L}$

- Notation: $L = H^0(X, \mathcal{O}_X)$, $L \subset L' \subset L^{1/p}$, $Y = (X \otimes_L L')^{\nu}_{\mathrm{red}}$
- Main Formula: $\phi^* K_X - K_Y \sim (p-1)(\mathfrak{F}_{L'/L} + \mathfrak{M}_{L'/L})$

Theorem (Properties of $\mathfrak{F}_{L'/L}$ and $\mathfrak{M}_{L'/L}$)

Fixed part $\mathfrak{F}_{L'/L}$

- $\mathfrak{F}_{L'/L} = 0 \iff (X \otimes_L L')_{\mathrm{red}}$ *is normal*
- $\operatorname{Supp} \mathfrak{F}_{L'/L} = \operatorname{Supp}$ *of conductor divisor of* $Y \to (X \otimes_L L')_{\mathrm{red}}$

Movable part $\mathfrak{M}_{L'/L}$

- $\mathfrak{M}_{L'/L} = 0 \iff X \otimes_L L'$ *is reduced*
- *Let* $X \dashrightarrow V$ *be the Stein factorization of the rational map* $X \dashrightarrow \mathrm{Im}(g_{\mathfrak{M}})^p$ *induced by* $\mathfrak{M}$

$$\begin{tikzcd}
Y \arrow[r] \arrow[rr, bend left, "F_Y"] \arrow[d, dashed, "g_{\mathfrak{M}}"] & X \arrow[r] \arrow[d, dashed, "f"] & Y^p \arrow[d, dashed, "g_{\mathfrak{M}}"] \\
\mathrm{Im}(g_{\mathfrak{M}}) & V \arrow[r] & \mathrm{Im}(g_{\mathfrak{M}})^p
\end{tikzcd}$$

Universal properties of the map induced by $\mathfrak{M}_{L'/L}$

$$
\begin{tikzcd}
Y \arrow[rr] \arrow[rrr, bend left, "F_Y"] \arrow[d, dashed] & & X \arrow[r] \arrow[d, dashed, "f"] & Y^p \arrow[d, dashed, "g_{\mathfrak{M}}"] \\
W \arrow[r] & V \otimes_L L' \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & \mathrm{Im}(g_{\mathfrak{M}})^p \\
& \operatorname{Spec} L' \arrow[r] & \operatorname{Spec} L &
\end{tikzcd}
$$

$f\colon X \dashrightarrow V$ = Stein factorization of $X \dashrightarrow \mathrm{Im}(g_{\mathfrak{M}})$
W = normalization of $(V \otimes_L L')_{\mathrm{red}}$

Theorem (Properties of $\mathfrak{M}$, continued)

- *$X_\eta \otimes_{k(V)} k(W)$ is reduced (where X_η = generic fiber of f)*
- *$\mathfrak{M}$ is the pullback of $\mathfrak{M}_W$*
- *If $X \dashrightarrow Z$ is a rational map whose generic fiber stays reduced after base changing to $k((Z \otimes_L L')_{\mathrm{red}})$, then $X \dashrightarrow Z \dashrightarrow V$*

Defining $\mathfrak{F}$ and $\mathfrak{M}$

Notation

$$
\begin{tikzcd}
Y \arrow[r, "\nu"] \arrow[drrr, "g"'] & Z = (X \otimes_L L')_{\mathrm{red}} \arrow[r, "\mathrm{red}"] & X \otimes_L L' \arrow[r] \arrow[d] & X \arrow[d] \\
 & & L' \arrow[r] & L
\end{tikzcd}
$$

- From P–W have $\mathcal{F} \hookrightarrow g^*\mathcal{T}_{L'/L} \cong \mathcal{O}_Y^{\oplus s}$
- Define $\mathcal{F}' =$ saturation of $\mathcal{F} = \mathcal{T}_{Y/X}$ in $g^*\mathcal{T}_{L'/L}$
- Define $\mathfrak{F} \in |\det(\mathcal{F}'/\mathcal{F})|$

Key idea: look at Ω instead of $\mathcal{T} = \Omega^\vee$

- We show $\mathcal{F}' = (\nu^*\Omega_{Z/X})^\vee$
- Can interpret $\mathfrak{F}$ in terms of $\Omega_{Y/Z}$ via cotangent sequence of $Y \to Z \to X$
- We define $\mathfrak{M}_{L'/L} \subset |\det(\nu^*\Omega_{Z/X})^{\vee\vee}| = |-\det \mathcal{F}'|$
- Then $\mathfrak{F} + \mathfrak{M} \sim -\det \mathcal{F}$

Movable part $\mathfrak{M} \subset |\det(\nu^*\Omega_{Z/X})^{\vee\vee}|$

$$Y \xrightarrow{\nu} Z = (X \otimes_L L')_{\mathrm{red}} \xrightarrow{\mathrm{red}} X \otimes_L L' \longrightarrow X$$

- $\mathcal{F}' = (\nu^*\Omega_{Z/X})^\vee$ = saturation of $\mathcal{F} = \mathcal{T}_{Y/X}$ in $g^*\mathcal{T}_{L'/L} \cong \mathcal{O}_Y^{\oplus s}$

I = ideal sheaf of $Z \subset X \otimes_L L'$

$$I/I^2 \longrightarrow \Omega_{X\otimes_L L'/X} \otimes \mathcal{O}_Z = \Omega_{L'/L} \otimes \mathcal{O}_Z \cong \mathcal{O}_Z^{\oplus s} \longrightarrow \Omega_{Z/X} \longrightarrow 0$$

Pulls back to

$$\nu^* I/I^2 \longrightarrow g^*\Omega_{L'/L} \cong \mathcal{O}_Y^{\oplus s} \longrightarrow \nu^*\Omega_{Z/X} \longrightarrow 0$$

Y normal $\implies$ $(\nu^*\Omega_{Z/X})^{\vee\vee}|_U$ locally free on an open U with $\mathrm{codim}_Y(Y \setminus U) \geq 2$

Definition

The movable linear system $\mathfrak{M}_{L'/L} \subset |\det((\nu^*\Omega_{Z/X})^{\vee\vee})|$ of Weil divisors on Y is defined by the map

$$\bigwedge^r g^*\Omega_{L'/L}|_U \twoheadrightarrow \bigwedge^r (\nu^*\Omega_{Z/X})^{\vee\vee}|_U$$

where $r = \mathrm{rank}(\nu^*\Omega_{Z/X}) = \mathrm{rank}\,\mathcal{F}$.

$\mathfrak{M}_{L'/L} = 0 \iff X \otimes_L L'$ is reduced

- $\mathcal{F}' = (\nu^*\Omega_{Z/X})^\vee$ = saturation of $\mathcal{F}$ in $g^*\mathcal{T}_{L'/L}$
- $\mathfrak{M}_{L'/L} \subset |\det((\nu^*\Omega_{Z/X})^{\vee\vee})|$ defined by $\bigwedge^r g^*\Omega_{L'/L}|_U \twoheadrightarrow \bigwedge^r (\nu^*\Omega_{Z/X})^{\vee\vee}|_U$

Proposition

$\mathfrak{M}_{L'/L} = 0 \iff X \otimes_L L'$ is reduced ($\iff X \otimes_L L'$ is R_0 and S_1)

- Easy direction: $X \otimes_L L'$ is $R_0 \implies \mathfrak{M}_{L'/L} = 0$

Proof.

$\nu^* I/I^2 \longrightarrow g^*\Omega_{L'/L} \longrightarrow \nu^*\Omega_{Z/X} \longrightarrow 0$ dualizes to

$$0 \longrightarrow \mathcal{F}' \longrightarrow g^*\mathcal{T}_{L'/L} \longrightarrow (\nu^* I/I^2)^\vee$$

$X \otimes_L L'$ is $R_0 \implies I/I^2$ is torsion $\implies (\nu^* I/I^2)^\vee = 0 \implies$
$-\det \mathcal{F}' = \det(\nu^*\Omega_{Z/X})^{\vee\vee} = 0 \implies \mathfrak{M}_{L'/L} \subset |\det(\nu^*\Omega_{Z/X})^{\vee\vee}|$ is 0 $\square$

- Other direction is harder

$\mathfrak{M}_{L'/L} = 0 \iff X \otimes_L L'$ is reduced

$$Y \xrightarrow{\nu} Z = (X \otimes_L L')_{\mathrm{red}} \xrightarrow{\mathrm{red}} X \otimes_L L' \longrightarrow X$$

To show $\implies$:

- Define the smallest subextension $L \subset E \subset L'$ detecting all the non-reducedness of $X \otimes_L L'$ ("essential part" of base change)

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \tilde{Y} := (X \otimes_L E)^{\nu}_{\mathrm{red}} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} L' \arrow[r] & \operatorname{Spec} E \arrow[r] & \operatorname{Spec} L \end{tikzcd}$$

  $\tilde{Y} \otimes_L L'$ is reduced, and E satisfies certain universal properties
- Construct E by taking the quotient of L' by a foliation
- This foliation on $\operatorname{Spec} L'$ comes from pushing forward $\Omega^{\vee}_{Z/X}$ from Z

Thank you!

Thank you for your attention!