1 Infinitesimal Generators

1.1 Let \( \{T(t) : t \geq 0\} \) is a semi-group of bounded operators in Banach space \( \mathcal{X} \), i.e., it satisfies that \( T(t)T(s) = T(t+s) \) for all \( s, t > 0 \) and \( T(0) = I \). Let \( f(t) = \ln \|T(t)\| \). Suppose that \( f(t) \) is bounded on \( [0, a] \), show that

\( f(t) \) is sub-additive, i.e., \( f(t+s) \leq f(t) + f(s) \) for all \( t, s > 0 \).

\[ \lim_{t \to \infty} \frac{1}{t} f(t) = \inf_{t > 0} \frac{1}{t} f(t). \]

**Proof.**

(1) \( f(t+s) = \ln \|T(t+s)\| = \ln \|T(t)T(s)\| \leq \ln(\|T(t)\| \|T(s)\|) = \ln \|T(t)\| + \ln \|T(s)\| = f(t) + f(s); \]

(2) It is not difficult to see that \( f(t) \) is bounded on any finite interval \( [0, s] \). Suppose that \( f(t) \) is bounded by \( M_s \) on \( [0, s] \). Fix \( s \). Any \( t \) can be written as \( t = ns + r \), where \( n \) is an integer and \( 0 \leq r < a \). Then we have from subadditivity of \( f \) that

\[ f(t) \leq nf(s) + f(r) \leq nf(s) + M_s. \]

Divide it by \( t \),

\[ \frac{f(t)}{t} \leq \frac{n}{ns + r} f(s) + \frac{M_s}{t} \leq \frac{f(s)}{s} + \frac{M_s}{t}. \]

Let \( t \to \infty \),

\[ \limsup_{t \to \infty} \frac{f(t)}{t} \leq \frac{f(s)}{s}. \]

Take infimum on the right-hand side,

\[ \liminf_{t \to \infty} \frac{f(t)}{t} \geq \inf_{s > 0} \frac{f(s)}{s}. \]

Notice that it holds trivially

\[ \liminf_{t \to \infty} \frac{f(t)}{t} \geq \inf_{s > 0} \frac{f(s)}{s}. \]

The proof is now complete. \( \Box \)

1.2 Let \( \{T(t) : t \geq 0\} \) is a semigroup of bounded operators, such that \( T(0) = I \) and strong continuity at \( t = 0 \), i.e.,

\[ \lim_{t \to 0^+} T(t) = I \].

Show that the semi-group is strongly continuous.

**Proof.** We shall show that \( t \mapsto T(t)x \) is continuous for all \( x \in \mathcal{X} \). It is easy to show right strong continuity.

\[ \lim_{t \to t_0^+} \|T(t)x - T(t_0)x\| = \lim_{t \to t_0^+} \|T(t_0)T(t-t_0)x - x\| \leq \|T(t_0)\| \lim_{t \to t_0^+} \|T(t-t_0)x - x\| = 0. \]

To prove the left strong continuity, it suffices to show that \( \|T(t)\| \) to be uniformly bounded when \( t \) is near \( t_0 \). In fact, it holds that \( \|T(t)\| \leq Me^{\omega t} \) for some \( M \) and \( \omega \). Refer to the text before Lemma 7.1.6. \( \Box \)

1.3 Let \( \{T(t) : t \geq 0\} \) is a semigroup of bounded operators on Hilbert space \( \mathcal{H} \) and satisfies \( T(0) = I \) and weak continuity at \( t = 0 \). Show that the semigroup is strongly continuous.

**Proof.** Since \( T(t)x \to x \), the uniform boundedness theorem tells us that \( \|T(t)x\| \) is uniformly bounded in a neighbourhood of \( t = 0 \). Again by the uniform boundedness principle, it holds that \( \|T(t)\| \) is uniformly bounded near \( t = 0 \). It is then easy to see that for a fixed \( x_0 \in \mathcal{X}, x(t) = T(t)x_0 \) is bounded on any compact interval of \( t \).

Suppose that \( 0 \leq a < t < b < \xi - \epsilon < \xi \), where \( \epsilon > 0 \). Since \( x(\xi) = T(\xi)x_0 = T(t)T(\xi-t)x_0 = T(t)x(\xi-t) \), we have that

\[ (b-a)x(\xi) = \int_a^b x(\xi) d\eta = \int_a^b T(t)x(\xi-t) d\eta. \]
In particular, there exists (3)

Proof.

that the following three conditions are equivalent:

(1) $\lim_{t \to 0^+} |T(t) - I| = 0$;
(2) $A \in L(\mathcal{X})$ and $T(t) = \exp(tA)$.

Proof. (3)⇒(1): It follows easily from series manipulation that $A_t x \to Ax$ for all $x$ as $t \to 0$.
(1)⇒(2): Since $A_t x \to x$ as $t \to 0$, by uniform boundedness principle, $A_t$ is uniformly bounded, say by $M$, in a small neighbourhood of $t = 0$. Then $\|T(t) - I\| \leq Mt \to 0$ as $t \to 0$.
(2)⇒(3): It is easy to verify that

$$
\lim_{s \to 0^+} \frac{1}{s} \int_r^{r+s} T(t) dt = T(r)
$$

In particular, there exists $\delta$ such that

$$
\left\| \frac{1}{s} \int_0^s T(s) ds - I \right\| < 1
$$

for all $0 < t < \delta$, then $\frac{1}{t} \int_0^t T(s) ds$ is invertible for $t \in (0, \delta)$. Now,

$$
\frac{1}{s} \int_r^{r+s} T(t) dt - \frac{1}{s} \int_0^s T(t) dt = \frac{1}{s}(T(s) - I) \int_0^r T(t) dt
$$

(because $\int_r^{r+s} - \int_0^s = \int_0^{r+s} - \int_0^r$). It follows that for $t \in (0, \delta)$,

$$
\frac{1}{s}(T(s) - I) = \left( \frac{1}{s} \int_r^{r+s} T(t) dt - \frac{1}{s} \int_0^s T(t) dt \right) \left( \int_0^r T(t) dt \right)^{-1}
$$
The right-hand side tends to \((T(r) - I)(\int_0^r T(t)dt)^{-1}\), so the left-hand side \(A_s\) converges to some bounded linear operator when \(s \to 0\). It is obvious that this limit operator must be \(A\). Taking limit \(s \to 0\) in (2), we obtain that

\[
T(r) - I = A \int_0^r T(s)ds
\]

Iterated substitution gives

\[
T(r) = I + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \frac{A^{n+1}}{n!} \int_0^r (r-t)^n T(t)dt
\]

Let \(n \to \infty\), we see that \(T(r) = \exp(rA)\) for all \(r \geq 0\).

1.5 Let \(X = C_0[0, \infty) = \{f \in C[0, \infty) : \lim_{x \to +\infty} f(x) = 0\}, \|f\| = \sup |f(s)|\). Define on \(X\) a linear operator

\[
T(t) : a(\cdot) \mapsto a(t + \cdot).
\]

Show that \(\{T(t) : t \geq 0\}\) is a strongly contraction semigroup on \(X\).

**Proof.** It is obvious that \(T(t+s) = T(t) + T(s)\) and \(T(0) = I\). Now we show that \(\|T(t)a - T(t_0)a\| \to 0\). This is because \(\|T(t)a - T(t_0)a\| = \sup_a |a(t+s) - a(t_0 + s)|\) and \(a\) is uniformly continuous. Finally, it is obvious that \(\|T(t)\| \leq 1\) for all \(t \geq 0\).

1.6 Let \(X = L^2(\mathbb{R})\), for \(x \in \mathbb{R}\) and \(y \in \mathbb{R_+}\), define

\[
((T(y)f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi)d\xi, \quad y > 0,
\]

\[
T(0)f = f.
\]

Show that \(\{T(y) : y \geq 0\}\) is a strongly continuous semigroup on \(X\) and \(\|T(y)\| = 1\). *(Remark. The integral gives a harmonic function on the upper plane with boundary value \(f)\)*

**Proof.** First we show that \(T(y)f \in L^2(\mathbb{R})\). Indeed, by Cauchy-Schwarz inequality,

\[
\int_{-\infty}^{\infty} \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi)d\xi \right|^2 dx \\
\leq \frac{1}{\pi^2} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} d\xi \right)^2 \left( \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} |f(\xi)|^2 d\xi \right) dx \\
= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} |f(\xi)|^2 d\xi dx \\
= \|f\|_2^2
\]

Hence \(\|T(y)\| \leq 1\). On the other hand,

\[
(T(y)\chi_{[-R,R]})(x) = \frac{1}{\pi} \left( \arctan \left( \frac{R-x}{y} \right) + \arctan \left( \frac{R+x}{y} \right) \right)
\]

and thus

\[
\frac{\|T(y)\chi_{[-R,R]}\|_2^2}{\chi_{[-R,R]}^2} \geq \frac{1}{R} \int_{-R/2}^{R/2} \frac{1}{\pi} \left( \arctan \left( \frac{R-x}{y} \right) + \arctan \left( \frac{R+x}{y} \right) \right)^2 dx \\
\geq \frac{1}{R} \int_{-R/2}^{R/2} \left( \frac{2}{\pi} \arctan \frac{R}{2y} \right)^2 dx
\]

3
Let \( f \) allows us to extend this result to be given. Since \( \Delta u = 0, \ y > 0 \) because \( S \)

when \( f \in \mathcal{D}(\mathbb{R}) \), Notice that \( T(y)f \) is exactly \( u(\cdot, y) \) that satisfies

\[
\begin{align*}
\Delta u &= 0, \quad y > 0 \\
u(x, 0) &= f(x, 0)
\end{align*}
\]

It is then obvious that \( T(t+s) = T(t) + T(s) \) for \( f \in \mathcal{D}(\mathbb{R}) \), which can be extended to the entire \( L^2(\mathbb{R}) \) easily because \( \mathcal{D}(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \) and \( \| T(y) \| = 1 \).

Now we show \( \| T(y)f - f \| \to 0 \) as \( y \to 0^+ \). It suffices to show this for \( f \in C^\infty_0(\mathbb{R}) \) and density of test functions allows us to extend this result to \( L^2(\mathbb{R}) \). First we show that \( T(y)f \to f \) uniformly pointwise as \( y \to 0^+ \). Let \( \epsilon > 0 \) be given. Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \epsilon \) whenever \( |x - y| < \delta \). Then

\[
|\langle T(y)f \rangle(x) - f(x)\rangle = \int_{\mathbb{R}} \frac{y}{(x - \xi)^2 + y^2} (f(\xi) - f(x))d\xi
\]

\[
\leq \int_{|x - \xi| < \delta} \frac{y}{(x - \xi)^2 + y^2} d\xi + \int_{|x - \xi| > \delta} \frac{y}{(x - \xi)^2 + y^2} d\xi
\]

\[
= I + J,
\]

where

\[
I \leq \epsilon \int_{\mathbb{R}} \frac{y}{(x - \xi)^2 + y^2} d\xi = \epsilon
\]

and

\[
J \leq 2\| f \|_{\infty} \int_{|x - \xi| > \delta} \frac{y}{(x - \xi)^2 + y^2} d\xi
\]

\[
= 2\| f \|_{\infty} \left( \pi - 2\arctan \frac{\delta}{y} \right) \to 0
\]

uniformly w.r.t. \( x \) as \( y \to 0 \). Hence \( T(y)f \to f \) uniformly w.r.t. \( x \). Suppose that supp \( f \in [-K, K] \). Now,

\[
\| T(y)f - f \| \leq \int_{|x| \leq K} |\langle T(y)f \rangle(x) - f(x)\rangle| + \int_{|x| \geq K} |\langle T(y)f \rangle(x)\rangle|^2 dx
\]

\[
\leq \| T(y)f - f \|_{\infty} \cdot 2K + \int_{|x| \geq K} |\langle T(y)f \rangle(x)\rangle|^2 dx
\]

The first term goes to 0 as \( y \to 0^+ \). For the second term, we have

\[
\int_{|x| \geq K} |\langle T(y)f \rangle(x)\rangle|^2 dx \leq \frac{1}{K} \int_{|\xi| \leq K} |f(\xi)|^2 \int_{|x| \geq K} \frac{y}{(x - \xi)^2 + y^2} dxd\xi
\]

\[
\leq \frac{1}{K} \int_{|\xi| \leq K} |f(\xi)|^2 \left( \pi - \arctan \frac{K - \xi}{y} - \arctan \frac{K + \xi}{y} \right) d\xi
\]

\[
\to 0 \text{ as } y \to 0^+ \text{ by Dominated Convergence Theorem.}
\]

Therefore we conclude that \( (T(y) - I)f \to 0 \) as \( y \to 0^+ \), whence the strong continuity condition is satisfied by Problem 1.2.

1.7 Let \( \{T(t) : t \geq 0\} \) be a strongly continuous semi-group on \( \mathcal{D} \). Suppose that \( x \in X, \ w-\lim_{t \to 0^+} \frac{1}{t}(T(t) - I)x = y \), show that \( x \in D(A) \) and \( y = Ax \).
1.10 Let $f \in X^*$ and $\lambda > 0$ large enough. It is easy to see that $e^{-\lambda t} f(T(t)x)$ has right derivative, which equals to $e^{-\lambda t} f(T(t)(y - \lambda x))$. The derivative is continuous in $t$, we have on integration

$$f(x) = -\int_0^\infty e^{-\lambda t} f(T(t)(y - \lambda x))dt,$$

for all $f \in X^*$. Therefore,

$$x = -\int_0^\infty e^{-\lambda t} T(t)(y - \lambda x) = -(\lambda I - A)^{-1}(y - \lambda x)$$

The conclusion follows immediately by multiplying $(\lambda I - A)$ on both sides.

1.8 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group on a Hilbert space $\mathbb{H}$. Suppose that $A$ is its generator and $T(t)$ is a normal operator for all $t \geq 0$. Show that $A$ is normal using Gelfand transform.

1.9 Prove Hille-Yosida-Phillips Theorem (Theorem 7.1.7): A densely-defined closed linear operator $A$ is an infinitesimal generator of some strongly continuous semigroup $\{T(t) : t \geq 0\}$ if and only if

1. There exists $\omega_0 > 0$ such that $(\omega_0, \infty) \subset \rho(A)$;
2. There exists $M > 0$ such that

$$\| \lambda I - A \| \leq \frac{M}{\lambda - \omega_0}, \quad n = 1, 2, \ldots$$

whenever $\lambda > \omega > \omega_0$.

Proof: The necessity has been proved in Lemma 7.1.6. The proof of sufficiency follows the same outline as in Theorem 7.1.5 by defining

$$B_\lambda = \lambda^2 (\omega_0 + \lambda - A)^{-1} - \lambda I$$

for $\lambda > 0$.

1.10 Let $\{T(t) : t \geq 0\}$ be a strongly continuous semi-group and $A$ its infinitesimal generator. Suppose that $\omega_0 \in \mathbb{R}$ satisfies $\{ \lambda : \Re \lambda > \omega_0 \} \subset \rho(A)$. Show that

1. The set $\{R_\lambda(A)x : x \in D(A)\}$ is dense in $D(A)$, where $\Re \lambda > \omega_0$;
2. The range of $R_\lambda(A)^n$ is dense for all $n \geq 1$, where $\Re \lambda > \omega_0$;
3. $(D(A^n))$ is dense for all $n \geq 1$.

Proof: (1) Let $x \in D(A^2)$. It follows from $R_\lambda(A)(\lambda - A)x = x$ that $x \in R(R_\lambda(A)|_{D(A)})$. Hence $D(A^2) \subseteq R(R_\lambda(A)|_{D(A)})$. The conclusion follows immediately from the density of $D(A^2)$.

(2) It follows from $R_\lambda(A)(\lambda - A)x = x$ that $x \in R(R_\lambda(A))$, that is, $D(A) \subseteq R(R_\lambda(A))$ and $R(R_\lambda(A))$ is therefore dense. Now, it $R_\lambda(A)^n(\lambda - A)^nx = x$ for all $x \in D(A^n)$, whence it follows that $D(A^n) \subseteq R(R_\lambda(A)^n)$. Part (3) shows that $D(A^n)$ is dense, and hence $R(R_\lambda(A)^n)$ is dense.

(3) This statement actually hold for any strongly continuous semi-group with no further assumptions. To see this, let $\phi$ be any function in $C_0^\infty[0, 1]$ and define for any $x$

$$x_\phi = \int_0^1 \phi(t)T(t)x dt.$$

Then

$$\frac{T(h) - 1}{h} x_\phi = \frac{1}{h} \int_0^1 \left( \int_0^t \phi'(s) ds \right) (T(t + h) - T(t))x dt.$$
1.1 Let $f$ and then it follows that Note that condition, it suffices to show that the second term satisfies 3 with initial value $x_0$. Proof.\[C^1([-1,1])\]

Since \(h\to 0^+\) and \(\phi'\) is bounded, the limit of the right-hand side exists as \(h\to 0^+\) and equals to
\[
-\int_0^1 \phi'(s)T(s)xds.
\]
Hence $x_0 \in D(A)$ and
\[
Ax_0 = -\int_0^1 \phi'(s)T(s)xds.
\]
Now it is clear that $x_0 \in D(A^n)$ for all $n$. We can choose a sequence \(\{\phi_j\} \subset C^\infty[0,1]\) such that \(\phi_j \geq 0\), \(\int_0^1 \phi_j = 1\) and \(\text{supp} \phi_j \to 0\). It is not difficult to see that \(x_{\phi_j} \to x\). Hence $D(A^n)$ is dense.  

1.11 Let \[\{T(t) : t \geq 0\}\] be a strongly continuous semi-group and $A$ its infinitesimal generator. Suppose that $f \in C^1([0,\infty);\mathcal{X})$. Show that the differential equation of operators
\[
\frac{dx(t)}{dt} = Ax(t) + f(t),
\]
\[x(0) = x_0 \in D(A)
\]
has a unique solution in $C([0,1];D(A)) \cap C^1([0,1),\mathcal{X})$, which is given by
\[
x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds.
\]
Proof. The first term $T(t)x_0$ on the right of (5) satisfies the homogeneous differential equation and the initial condition, it suffices to show that the second term satisfies 3 with initial value 0.
\[
\int_0^t T(t-s)f(s)ds = \int_0^t T(t-s) \left( f(0) + \int_0^s f'(r)dr \right) ds
\]
\[
= \left( \int_0^t T(t-s)ds \right) f(0) + \int_0^t f'(r) \left( \int_r^t T(t-s)ds \right) dr
\]
Note that
\[
T(t) - T(r) = \int_r^t AT(s)ds,
\]
it follows that
\[
A \int_r^t T(t-s)ds = T(t-r) - I,
\]
and then
\[
A \int_0^t T(t-s)f(s)ds = (T(t) - I)f(0) + \int_0^t f'(r)(T(t-r) - I)dr
\]
2 Examples of Infinitesimal Generators

2.1 Let \( X \)

On the other hand,

\[
\frac{d}{dt} \int_0^t T(t-s)f(s)ds = T(t)f(0) + \int_0^t T(s)f'(t-s)ds.
\]

Comparing (6) and (7), we see that

\[
\frac{d}{dt} \int_0^t T(t-s)f(s)ds = A \int_0^t T(t-s)f(s)ds + f(t).
\]

It is clear that the initial value of the second term of (5) is 0. Hence \( x(t) \) given in (5) is a solution to the differential equation indeed. The continuity of \( x' (t) \) can be easily concluded from (7) using the continuity of \( f' \). It is clear that \( x(t) \in D(A) \).

In fact, if \( x(t) \in C(\mathbb{R}^+; D(A)) \cap C^1(\mathbb{R}^+; \mathcal{X}) \) is a solution to (3) and (4), then

\[
\frac{d}{ds} T(t-s)x(s) = -T'(t-s)x(s) + T(t-s)x'(s) = -T(t-s)Ax(s) + T(t-s)x'(s) = T(t-s)f(s).
\]

Integrate on both sides, we obtain that

\[
x(t) - T(t)x(0) = \int_0^t T(t-s)f(s)ds,
\]

which is exactly (5). The uniqueness of the solution is proved.

\[
= T(t)f(0) - f(t) + \int_0^t f'(r)T(t-r)dr
\]

2 Examples of Infinitesimal Generators

2.1 Let \( \mathcal{X} = \{ f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} c_n z^n, \| f \|^2 = \sum |c_n|^2 < \infty \} \), where \( \mathbb{D} \) is the open disc in the complex plane. Define on \( \mathcal{X} \)

\[
(T(t)f)(z) = \sum_{n=0}^{\infty} (n+1)^{-t} c_n z^n.
\]

Show that \( \{ T(t) : t \geq 0 \} \) is a strongly continuous semi-group of positive self-adjoint operators. Find its infinitesimal generator \( A \) and show that \( \ln \frac{1}{n+1} (n \geq 1) \) are eigenvalues of \( A \).

**Proof:** It is obvious that \( \| T(t) \| \leq 1, T(t+s) = T(t)T(s) \) and \( T(0) = I \). Now we show that \( T(t)f \to f \) strongly for all \( f \in \mathcal{X} \). Given \( \epsilon > 0 \) and \( f = \sum_{n=0}^{\infty} c_n z^n \), choose \( N \) big enough such that \( \sum_{n>N} |c_n|^2 < \frac{\epsilon^2}{2} \). Choose \( t \) small enough such that \( 1 - \left( \frac{1}{N+1} \right)^t < \frac{\epsilon^2}{\| f \|^2} \). Then

\[
\| f - T(t)f \|^2 = \sum_{n=0}^{\infty} \left( 1 - \frac{1}{(n+1)^t} \right)^2 |c_n|^2 + \sum_{n=N+1}^{\infty} \left( 1 - \frac{1}{(n+1)^t} \right)^2 |c_n|^2
\]

\[
\leq \frac{\epsilon^2}{2\| f \|^2} \cdot \| f \|^2 + \sum_{n=N+1}^{\infty} |c_n|^2
\]

\[
\leq \frac{\epsilon^2}{2} + \epsilon^2 = \epsilon^2,
\]

that is, \( \| f - T(t)f \| \leq \epsilon \) when \( t \) is sufficiently small. Therefore \( \{ T(t) : t \geq 0 \} \) is a strongly continuous semi-group. It is straightforward to verify that \( T(t) \) is positive and self-adjoint (note that the inner product \( \langle \sum c_n z^n, \sum d_n z^n \rangle = \sum c_n \overline{d_n} \)).
Define a linear operator $A$ as

$$Af = \sum_{n=0}^{\infty} c_n \ln \frac{1}{n+1} \cdot x^n$$

on

$$D(A) = \left\{ f = \sum c_n z^n \in \mathcal{A} : \sum |c_n|^2 \ln^2 (n+1) < \infty \right\}.$$

It is clear that $D(A)$ is dense because $x^n \in D(A)$ for all $n$. We claim that $A$ is closed. Suppose that $f_n \to f$, $A f_n \to g$, $f_n = \sum c_n z^n$ and $g = \sum d_n z^n$. Since $A f_n \to g$,

$$\sum_{k=0}^{\infty} |c_{nk} \ln \frac{1}{k+1} - d_k|^2 \to 0,$$

or,

$$\sum_{k=1}^{\infty} \left| c_{nk} - \frac{d_k}{\ln (k+1)} \right|^2 \to 0,$$

(because $d_0$ must be 0) which implies that

$$\sum_{k=0}^{\infty} \left| c_{nk} - \frac{d_k}{\ln (k+1)} \right|^2 \to 0.$$

Comparing with $f_n \to f$, or, equivalently,

$$\sum_{k=0}^{\infty} |c_{nk} - d_k|^2 \to 0.$$

we obtain that

$$d_n = c_n \ln \frac{1}{n+1}$$

for all $n \geq 0$, that is, $f \in D(A)$ and $g = Af$. Therefore $A$ is a densely-defined closed operator.

Next we show that $A$ generates a contraction semigroup. For $\lambda > 0$, it is easy to verify that $\lambda I - A$ is injective, and $\|\lambda f - Af\| \geq \lambda \|f\|$, thus $\lambda I - A$ is invertible and $\|R_\lambda(A)\| \leq \lambda^{-1}$. By Hille-Yosida Theorem we know that $A$ generates a contraction semigroup.

Now, to show that $A$ is the infinitesimal generator of $\{T(t)\}$, it suffices to show that $A f \to Af$ on $D(A)$. Let $f \in D(A)$, $f = \sum c_n z^n$, then

$$\|A f - Af\|^2 = \sum |c_n|^2 \left| \frac{(n+1)^{-t} - 1}{t} - \ln \frac{1}{n+1} \right|^2$$

$$= \sum |c_n|^2 \left| e^{t \ln \frac{1}{n+1} - 1} - \frac{1}{n+1} \right|^2$$

$$\leq t \sum |c_n|^2 \ln^2 \frac{1}{n+1} \to 0$$

as $t \to 0^+$, where we used $e^x - 1 \leq x + x^2$ for all $x \leq 1$. \qed

2.2 Let $\mathcal{A} = L^2(-\pi, \pi)$. Define

$$(T(t)f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\theta - \xi, t)f(\xi) d\xi, \quad t > 0$$

$$T(0)f = f,$$

where the integral kernel $G(\theta, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 t} \cos n\theta$. Show that $\{T(t) : t \geq 0\}$ is a strongly continuous semi-group. Is it a contraction semi-group?
Proof. The procedure is similar to Exercise 7.1.6, based on the following properties of $G(\theta, t)$:

1. $G(\theta, t)$ is continuous;
2. $G(\theta, t) \geq 0$ for all $t$;
3. For every $\delta$, $K(x, t) \to 0$ uniformly on $\delta < |x| < \pi$.

We shall prove the properties later. For now let us assume their correctness. For simplicity let the kernel $G(\theta, t)$ absorb the normalisation coefficient $\frac{1}{\sqrt{\pi}}$. Using the same trick as in 7.1.6, we obtain that $\|T(t)\| \leq 1$ for all $t > 0$.

It is easy to verify that $T(t+s) = T(t)T(s)$ (because $e^{-n^2(t+s)} = e^{-n^2t}e^{-n^2s}$) and $T(0) = I$. To show the right strong convergence at $t = 0$, we can assume that $f \in C^1_0(S^1)$. Using uniform continuity of $f$ and Property (3) of the kernel $G(\theta, t)$, it is easy to show that $\|T(y)f - f\| \to 0$. We therefore conclude that $\{T(t)\}$ is a contraction semi-group.

To see Property (1), just notice that the sum of continuous functions is uniformly convergent. To see (2) and (3), apply Poisson’s Summation formula

$$
\sum_{k=-\infty}^{\infty} g(x + 2k\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{g}(n)e^{in\pi x}
$$

to

$$
g(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}
$$

(note that $\hat{g}(\xi) = e^{-\xi^2 t}$), we obtain that

$$
G(x, t) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(x+2k\pi)^2}{4t}},
$$

whence Property (2) becomes obvious. Property (3) follows from integral approximation of $G(x, t)$,

$$
G(\delta, t) \leq \frac{1}{\sqrt{\pi t}} \int_{\delta}^{\infty} e^{-\frac{x^2}{4t}} \, dx = 1 - \Phi\left(\frac{\delta}{2\sqrt{t}}\right) \to 0.
$$

Remark. In fact, $T(t)f$ gives the solution $u(. , t)$ to the heat equation on a circle $S^1$

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \theta^2}, \quad t > 0, \theta \in (-\pi, \pi)
$$

$$
u(\pi, t) = u(-\pi, t)
$$

$$
u(\theta, 0) = f(\theta), \quad \theta \in (-\pi, \pi)
$$

when $f \in H^2(S^1)$ such that $f(-\pi) = f(\pi)$.

2.3 Let $\mathcal{R} = C(\infty, \infty)$, the space of bounded uniformly continuous functions on $(-\infty, \infty)$. Define the linear operator $T(t)$ by

$$
(T(t)u)(s) = \begin{cases} u(s), & t = 0; \\
\sum_{n=0}^{\infty} \frac{(\lambda \mu)^n}{n!} u(s - n\mu), & t > 0. \end{cases}
$$

where $\lambda, \mu > 0$. Show that $\{T(t) : t \geq 0\}$ is a strongly continuous contraction semi-group, and its infinitesimal generator is the difference operator $A$:

$$
(Au)(s) = \lambda(u(s - \mu) - u(s)).
$$
Proof. It is easy to see that \( T(t)u \) is uniformly continuous (using the uniform continuity of \( f \)) and \( \|T(t)\| \leq 1 \). We have

\[
T_u(T_t(u))(s) = e^{-\lambda w} \sum_{n=0}^{\infty} \frac{(\lambda u)^{n} \sum_{m=0}^{\infty} \frac{(\lambda t)^{m}}{m!} u(s - m\mu - n\mu)}{n!} \]

\[
= e^{-\lambda(w + t)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(\lambda u)^{k-l}}{l!} \frac{(\lambda t)^{l}}{(k-l)!} u(s - k\mu) \\
= e^{-\lambda(w + t)} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k (w + t)^k u(s - k\mu) \\
= T_{t+w}(u)(s)
\]

It is trivial that \( T(0) = I \). Now we verify that \( \|T(t)u - u\|_{\infty} \to 0 \) strongly as \( t \to 0^+ \). Write

\[
(T(t)u - u)(x) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n}}{n!} u(x - n\mu) + (e^{-\lambda t} - 1) u(x)
\]

The second term goes to 0 uniformly because \( u \) is bounded. So does the first term, since

\[
\left| e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n}}{n!} u(x - n\mu) \right| \leq \|s\|_{\infty} e^{-\lambda t} (e^{-\lambda t} - 1) = \|u\|_{\infty} (1 - e^{-\lambda t}).
\]

Therefore \( \{T(t)\} \) is a strongly continuous contraction semi-group. Now,

\[
(A_t u)(x) = e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^n (n-1)!}{n!} u(x - n\mu) + \lambda e^{-\lambda t} u(x - \mu) + \frac{e^{\lambda t} - 1}{t} u(x).
\]

The second term goes to \( \lambda s(x - \mu) \) and the third term \( \lambda u(x) \), and both convergences are uniform. The first term goes to 0 uniformly, as

\[
\left| e^{-\lambda t} \sum_{n=2}^{\infty} \frac{\lambda^n (n-1)!}{n!} u(x - n\mu) \right| \leq \|s\|_{\infty} e^{-\lambda t} \left( \frac{e^{\lambda t} - 1}{t} - \lambda \right) = \|u\|_{\infty} \left( \frac{1 - e^{-\lambda t}}{t} - \lambda e^{-\lambda t} \right) \to 0.
\]

We conclude that

\[
s - \lim_{t \to 0^+} (A_t u)(s) = \lambda (u(s - \mu) - u(s))
\]

for all \( u \in C(-\infty, \infty) \). The limit function is in \( C(-\infty, \infty) \), too. The proof is now complete. \( \square \)

2.4 Let \( \mathcal{X} = C(\mathbb{R}^n, \mathbb{R}) \) and \( b \in \mathcal{X} \). Consider the following system of ODEs

\[
\frac{dx(t)}{dt} = f(x(t)), \quad x(0) = \xi,
\]

which is an autonomous system. For every \( \xi \in \mathbb{R}^n \) there exists a solution \( x(t, \xi) \), \( t \in \mathbb{R} \) such that \( x(t) \in \mathcal{X} \). Define the linear operator

\[
T(t) : f(\xi) \mapsto f(x(t, \xi)), \quad t \geq 0.
\]

Show that \( \{T(t) : t \geq 0\} \) is a strongly continuous semi-group. Let \( A \) be its generator, then \( C^1_0(\mathbb{R}^n; \mathbb{R}) \subseteq D(A) \) and whenever \( f \in C^1_0(\mathbb{R}^n; \mathbb{R}) \) it holds that

\[
(Af)(x) = \sum_{i=1}^{d} b^i(x) \frac{\partial f(x)}{\partial x_i}.
\]
2.5 Let $A$ be the infinitesimal generator of a contraction semi-group. Suppose that $B$ is a dissipative operator, with $D(B) \supset D(A)$, and

$$
\|Bu\| \leq a\|Au\| + b\|u\|,
$$

for some $0 < a < 1/2$, $b > 0$, and all $u \in D(A)$. Show that $A + B$ is a closed dissipative operator (defined on $D(A)$) and generates a contraction semi-group.

Proof. Since $A$ is the generator of a contraction semi-group, according to Hille-Yosida Theorem, $(\lambda - A)^{-1}$ exists for all $\lambda > 0$ and $\| (\lambda - A)^{-1} \| \leq 1/\lambda$. Thus $\|A(\lambda - A)^{-1}\| = \|\lambda(\lambda - A)^{-1} - I\| \leq 1$. For $u \in D(A)$,

$$
\|B(\lambda - A)^{-1}u\| \leq a\|A(\lambda - A)^{-1}u\| + b\|\lambda - A\|^{-1}u\| \leq \left(2a + \frac{b}{\lambda}\right)\|u\|.
$$

Thus for $\lambda$ sufficiently large, $\|B(\lambda - A)^{-1}\| < 1$ and $I - B(\lambda - A)^{-1}$ is invertible. Since $R(\lambda - A) = \mathcal{D}$ and

$$
\lambda - A - B = (I - B(\lambda - A)^{-1})(\lambda - A)
$$

we see that $R(\lambda - A - B) = \mathcal{D}$. It is easy to verify that $A + B$ is closed (Corollary 6.5.3) and dissipative (both $A$ and $B$ are dissipative). Hence $A + B$ generates a contraction semigroup.

Remark. The conclusion still holds if we know that $A + B$ is closed and dissipative, without assuming that $B$ is dissipative. This observation will be used in the proof of the next problem.

2.6 Let $A$ and $C$ be dissipative operators on Banach space $\mathcal{D}$. Suppose that there is a dense set $D$, $D \subset D(A)$, $D \subset D(C)$ so that

$$
\|(A - C)u\| \leq a(\|Au\| + \|Cu\|) + b\|u\|
$$

for some $0 < a < 1$, $b > 0$ and all $u \in D$. Show that

1. $A$ generates a contraction semigroup if and only if $C$ does.
2. $D(A|_D) = D(C|_D)$.

Proof. (1) It suffices to show that $R(\lambda - A)$ is dense for some $\lambda > 0$ if and only if $R(\mu - C)$ is dense for some $\mu > 0$. The proof is similar to Exercise 6.5.7. Let $B = C - A$ with $D(B) = D$ and define $T_\lambda = A + \lambda B$. Then $T_0 = A$, $T_1 = C$, $Au = T_\lambda u - \lambda Bu$ and $Cu = T_\lambda u + (1 - \lambda)Bu$. The inequality in the problem implies that

$$
\|Bu\| \leq a(\|T_\lambda u - \lambda Bu\| + \|T_\lambda u + (1 - \lambda)Bu\|) + b\|u\| \leq 2a\|T_\lambda u\| + a\|Bu\| + b\|u\|,
$$

or

$$
\|Bu\| \leq \frac{2a}{1 - a}\|T_\lambda u\| + \frac{b}{1 - a}\|u\| \quad (8)
$$

Let $0 \leq \lambda' \leq 1$. If $\frac{2a\lambda'}{1 - a} < \frac{1}{2}$, the preceding problem implies that $R(\lambda - T_{\lambda + \lambda'}) = R(\lambda - T_\lambda - \lambda' B)$ is dense for some $\lambda > 0$ if and only if $R(\mu - T_{\lambda'})$ is dense for some $\mu > 0$. Thus starting from $T_0 = A$ and applying this result finitely many times, we obtain the conclusion.

(2) It can be proved similarly by propagating the property from $T_0$ to $T_1$ in finitely many steps. Note that the inequality $(8)$ implies the equivalence of the graph norms with respect to $T_\lambda$ and $T_{\lambda + \lambda'}$. 

\[\square\]
3 One-Parameter Unitary Groups and Stone's Theorem

3.1 Let \{U(t) : t \in \mathbb{R}\} is a strongly continuous unitary group and \(D\) a dense set in \(X\) such that \(U(t)D \subset D\) for all \(t \in \mathbb{R}\). Suppose that \(U(t)\) is strongly differentiable on \(D\), i.e., \(U(t)x\) is differentiable with respect to \(t\) for all \(x \in D\). Show that \(-i\frac{dU(t)}{dt}|_{t=0}\) is essentially self-adjoint on \(D\), and its closure is the infinitesimal generator of the unitary group.

Proof. Let \(B = -i\frac{dU(t)}{dt}|_{t=0}\). Then for \(x, y \in D\),

\[
(Bx, y) = \lim_{t \to 0} -i \left( \frac{U(t) - I}{t} x, y \right) = \lim_{t \to 0} -i \left( x, \frac{U(-t) - I}{t} y \right) = \lim_{t \to 0} \left( x, -i \frac{U(-t) - I}{-t} y \right) = (x, By),
\]

which implies that \(B\) is symmetric. We can also establish that \(\frac{dU(t)}{dt}(t) = iAU(t)x = iU(t)Ax\). The rest follows similarly to part (3) and (4) of the next problem.

3.2 (Another proof of Stone's Theorem) Let \(\{U(t) : t \in \mathbb{R}\}\) is a strongly continuous unitary group.

(1) \(\forall f \in C_0^\infty(\mathbb{R}), \forall x \in X\), define

\[
x_f = \int_{-\infty}^{\infty} f(t)U(t)x dt,
\]

under Riemann sense. Let \(D\) be the set of all possible bounded linear combinations of \(x_f\)'s. Show that \(D\) is dense.

(2) For \(x \in D\), \(U(t)x\) is differentiable. Find

\[
\frac{dU(t)x}{dt} \bigg|_{t=0}
\]

(3) Define the operator \(A\) on \(D\) as

\[
Ax = -iU'(0)x,
\]

show that \(A\) is essentially self-adjoint.

(4) Let \(V(t) = e^{tA}\), show that \(V(t) = U(t)\).

Proof: (1) The integral exists because \(\|f(t)U(t)x\| \leq \|f\|_\infty \|x\|\), and the integral is over a compact set. Choose \(g \in C_0^\infty(\mathbb{R})\) with support in \([-1, 1]\) such that \(\int g = 1\) and let \(g_\epsilon(x) = \epsilon^{-1}g(x/\epsilon)\), then

\[
\|x - x_{g_\epsilon}\| \leq \int g_\epsilon(t)\|(U(t) - I)x\| \leq \sup_{|t| \leq \epsilon} \|(U(t) - I)x\| \to 0
\]
as \(t \to 0^+\) by strong continuity of \(\{U(t)\}\). Therefore \(D\) is dense.

(2) Let \(f \in C_0^\infty(\mathbb{R})\), we have

\[
\frac{U(t)x_f - U(t_0)x_f}{t - t_0} = \frac{1}{t - t_0} \left( \int_{-\infty}^{\infty} f(s)U(t + s)x ds - \int_{-\infty}^{\infty} U(t_0 + s)x ds \right)
\]

\[
= \frac{1}{t - t_0} \left( \int_{-\infty}^{\infty} f(r - t)U(r)x dr - \int_{-\infty}^{\infty} f(r - t_0)U(r)x dr \right)
\]

\[
= \int_{-\infty}^{\infty} \frac{f(r - t) - f(r - t_0)}{t - t_0} U(r)x dr
\]

for all \(t_0 \neq t\). Since \(f' \in C_0^\infty(\mathbb{R})\), the norm of the integrand on the right-hand side is dominated by \(\|f'\|_\infty \text{supp } f' \|x\|\) (using Lagrange's Mean-value Theorem), so we can apply Dominated Convergence Theorem, which yields that

\[
\lim_{t \to t_0} \frac{U(t) - U(t_0)}{t - t_0}x_f = -\int_{-\infty}^{\infty} f'(r - t_0)U(r)x dr,
\]

which is contained in \(D\) because \(f'(-t_0) \in C_0^\infty(\mathbb{R})\). In particular, \(U'(0)x_f = x_{f'}\).
3.4 Let \( U \) be an unitary operator on \( \mathcal{H} \) then the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = \bar{x}
\]
eexists and \( U \bar{x} = \bar{x} \).

Proof. Consider the spectrum decomposition of \( U \),
\[
U = \int_0^{2\pi} e^{i\theta} dF_\theta,
\]
then
\[
\frac{1}{N} \sum_{n=0}^{N-1} U^n = \int_0^{2\pi} \left( \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} \right) dF_\theta.
\]
Note that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} = \begin{cases} 1, & \theta = 0; \\ 0, & 0 < \theta < 2\pi. \end{cases}
\]

Denote the right-hand side by \( f(\theta) \). It is easy to see that
\[
E(\{1\})x = \int_0^{2\pi} f(\theta) dF_0 x.
\]

In fact, \( E(\{1\}) \) is the orthogonal projection of \( U \) onto the eigenvalue associated with eigenvalue 1. By Dominated Convergence Theorem,
\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n x - E(\{1\})x \right\|^2 = \int_0^{2\pi} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{in\theta} - f(\theta) \right|^2 d\|F_0 x\|^2 \to 0
\]
as \( N \to \infty \). Therefore
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = E(\{1\})x.
\]

It follows immediately from functional calculus that \( UE(\{1\})x = E(\{1\})x \).

3.5 Let \((\Omega, \mathcal{B}, \sigma)\) be a measure space of finite measure. Suppose that \( \{\Gamma_t : t \in \mathbb{R}\} \) is an ergodic group of measure-preserving transformations, show that

(1) \( \forall f, g \in L^2(\Omega, \mathcal{B}, \sigma) \),
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (f(\Gamma_t x), g) dt = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma \int_\Omega g d\sigma
\]

(2) Let \( A, B \in \mathcal{B} \) then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \sigma(\Gamma_t A \cap B) dt = \frac{1}{\sigma(\Omega)} \sigma(A) \sigma(B)
\]

**Proof.** (1) Since \( \{\Gamma_t\} \) is ergodic and \( \sigma(\Omega) < +\infty \),
\[
\lim_{T \to \infty} \int_0^T f(\Gamma_t x) dx = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma.
\]

It follows immediately from continuity of inner product that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (f(\Gamma_t x), g) dt = \lim_{T \to \infty} \left( \int_0^T f(\Gamma_t x) dx, g \right) = \frac{1}{\sigma(\Omega)} \int_\Omega f d\sigma \int_\Omega g d\sigma
\]

(2) In the previous part, let \( f = \chi_A \) and \( g = \chi_B \) and note that
\[
(\chi_A(\Gamma_t x), \chi_B) = \int_\Omega \chi_A(\Gamma_t x) \chi_B(x) dx = \sigma(\Gamma_t^{-1} A \cap \Gamma_B) = \sigma(\Gamma_{-t} A \cap \Gamma_B)
\]
and (with substitution \( t' = -t \))
\[
\frac{1}{T} \int_0^T \sigma(\Gamma_t A \cap B) dt = \frac{1}{-T} \int_0^{-T} \sigma(\Gamma_{-t'} A \cap B) dt'.
\]
3.6 Let $A$ and $B$ be positive self-adjoint operators. Suppose that $A + B$ is self-adjoint on $D(A) \cap D(B)$, $-A$, $-B$ and $-(A + B)$ can generate strongly continuous contraction semigroup, denoted by $\{T^A(t) : t \geq 0\}$, $\{T^B(t) : t \geq 0\}$ and $\{T^{A+B}(t) : t \geq 0\}$, respectively. Show that

$$T^{A+B}(t) = s^- \lim_{n \to \infty} \left(T^A \left( \frac{t}{n} \right) T^B \left( \frac{t}{n} \right) \right)^n$$

**Proof.** The proof follows the same outline as that of Trotter’s Product Formula (Theorem 7.3.14). Let $x \in D := D(A) \cap D(B)$, then

$$s^- \lim_{s \to 0^+} \frac{1}{s} (T^A(s)T^B(s)x - x) = s^- \lim_{s \to 0^+} \frac{1}{s} (T^A(s)x - x) + s^- \lim_{s \to 0^+} \frac{1}{s} T^A(s)(T^B(s)x - x) = -Ax - Bx$$

and

$$s^- \lim_{s \to 0^+} \frac{1}{s} (T^{A+B}(s)x - x) = -(A + B)x.$$ 

Let $T_s = \frac{1}{s}(T^A(s)T^B(s) - T^{A+B}(s))$, then $T_s x \to 0$ as $s \to 0^+$ and $T_s \to 0$ as $s \to +\infty$. Hence for any $x \in D$, $\|T_s x\|$ is bounded and thus $\|T_s\|$ is uniformly bounded by uniform boundedness principle. It follows that $T_s x \to 0$ uniformly on any compact set $K \subset D$.

Choose $K = \{T^{A+B}(r)x : r \in [-1, 1]\}$ (a continuous map maps a compact set to a compact set). We have

$$s^- \lim_{s \to 0^+} \frac{1}{s} (T^A(s)T^B(s) - T^{A+B}(s))T^{A+B}(r)x = 0$$

uniformly on $s \in [-1, 1]$.

From the interpolation

$$(T^A(s)T^B(s))^n - (T^{A+B}(s))^n x = \sum_{k=0}^{n-1} (T^A(s)T^B(s))^k(T^A(s)T^B(s) - T^{A+B}(s))(T^{A+B}(s))^{n-1-k}x$$

it follows that

$$\left\| \left( T^A \left( \frac{t}{n} \right) T^B \left( \frac{t}{n} \right) \right)^n x - \left( T^{A+B} \left( \frac{t}{n} \right) \right)^n x \right\|$$

$$\leq n \max_{|s| < t} \left\| \left( T^A \left( \frac{t}{n} \right) T^B \left( \frac{t}{n} \right) - T^{A+B} \left( \frac{t}{n} \right) \right) T^{A+B}(s)x \right\|$$

$$\leq |t| \max_{|s| < t} \left\| \frac{n}{t} \left( T^A \left( \frac{t}{n} \right) T^B \left( \frac{t}{n} \right) - T^{A+B} \left( \frac{t}{n} \right) \right) T^{A+B}(s)x \right\|$$

$$= |t| \max_{|s| < t} \left\| T_s T^{A+B}(s)x \right\| \to 0$$

as $n \to \infty$. Therefore we have established

$$s^- \lim_{n \to \infty} \left(T^A \left( \frac{t}{n} \right) T^B \left( \frac{t}{n} \right) \right)^n x = T^{A+B}(t)x$$

for all $x \in D$. Since $D$ is dense and the semi-groups are contractions, the limits holds for all $x \in \mathcal{H}$.

4 Markov Processes

No exercises.
5 Scattering Theory

No exercises.

6 Evolution Equations

No exercises.