1 The Electron Self-Energy

This problem is an assortment of questions checking claims made in PS sec. 7.1.
a) Verify PS 7.18, i.e. show that
\[ \int \frac{d^4 \ell}{(2\pi)^4} \left( \frac{1}{|\ell^2 - \Delta|^2} - \frac{1}{|\ell^2 - \Delta|^2} \right) = \frac{i}{(4\pi)^2} \log \left( \frac{\Delta_A}{\Delta} \right) \]
by reducing it to the standard integral PS 6.49.
b) Verify PS 7.20.
c) Assume that \( p^2 > (m_0 + \mu)^2 \). Show that at least one of the solutions found under b) is real and satisfies \( 0 < x < 1 \). This is the condition that the electron self-energy has a branch cut for momenta satisfying \( p^2 > (m_0 + \mu)^2 \).
d) Show that \( \delta F_1(0) + \delta Z_2 = 0 \). Start from PS 7.32. Then evaluate
\[ \int_0^1 dz (1 - 2z) \log \left( \frac{z^2}{(1 - z)^2 m^2 + z \mu^2} \right) \]
using integration by parts.

2 Bremsstrahlung at Higher Order

Consider the scattering of a charged fermion (such as an electron) from some other particle or from an external field. As in PS p 182 we write the scattering amplitude in the general form
\[ iM = \bar{u}(p')M_0(p', p)u(p) \]
This notation suppresses the dependence of the amplitude on the momenta and wave functions of all the other particles. Consider this scattering process accompanied by bremsstrahlung of a photon with momentum \( k^\mu \) from the charged fermion. For a photon with a very low energy \( \omega \) the amplitude \( \tilde{M} \) is given by PS eq. 6.22
\[ \tilde{M} = \bar{u}(p')M_0(p', p)u(p) \left[ e \left( \frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) \epsilon^*_\mu \right] \]
Here $\epsilon^\mu(k)$ is the polarization vector of the emitted soft photon. The purpose of this problem is to generalize this formulae to photons with energy of the same order as the electron energy. Again, we want a formula for scattering along with a photon $\tilde{M}$ in terms of the amplitude $M_0$ corresponding to scattering without Bremsstrahlung.

a) Conservation of the electromagnetic current or, equivalently, gauge invariance, requires that all amplitudes vanish when the polarization of an external photon is taken as $\epsilon^\mu = k^\mu$. Verify that the amplitude (??) satisfies this requirement.

b) Consider the amplitude $PS$ eq 6.20 for values of the photon energy which are not necessarily small. Show that this amplitude in general is inconsistent with current conservation.

c) The amplitude $PS$ eq 6.20 was derived by assuming that the radiated photon is emitted from the external legs associated with the charged particle. This assumption is correct at low energy but more generally there is a third class of diagrams, where the photon is emitted from the "gut" of the reaction process (This class of diagrams correspond to adding the soft photon to the filled circle in the figures in p 182, rather than to the external legs). The contribution from these diagrams take the general form

$$i\mathcal{M}_3 = \epsilon^*_\mu S^\mu$$

Determine the condition that current conservation imposes on $S^\mu$.

d) Expand your result from c) at low energies and solve for $S^\mu$.

e) The amplitude (??) for Bremsstrahlung is valid to order $O(1/\omega)$ in the energy $\omega$ of the emitted photon. Combine the results from above to write an improved formula, valid to $O(1)$. 

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