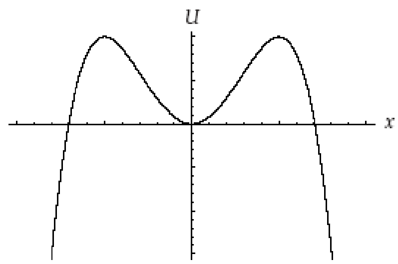
PHY 401 Fall 2004. Solutions to HW3.

2-52.

a) $F(x) = -\frac{dU}{dx} = -\frac{4U_0 x}{a^2}\left(1-\frac{x^2}{a^2}\right)$

b)



When $F = 0$, there is equilibrium; further when U has a local minimum (i.e. $dF/dx < 0$) it is stable, and when U has a local maximum (i.e. $dF/dx > 0$) it is unstable.

So one can see that in this problem $x = a$ and $x = -a$ are unstable equilibrium positions, and $x = 0$ is a stable equilibrium position.

c) Around the origin, $F \approx -\frac{4U_0 x}{a^2} \equiv -kx \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4U_0}{ma^2}}$

d) To escape to infinity from $x = 0$, the particle needs to get at least to the peak of the potential,

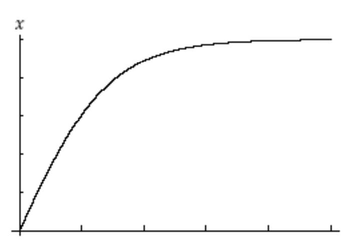
$$\frac{mv_{\min}^2}{2} = U_{\max} = U_0 \Rightarrow v_{\min} = \sqrt{\frac{2U_0}{m}}$$

e) From energy conservation, we have

$$\frac{mv^2}{2} + \frac{U_0 x^2}{a^2} = \frac{mv_{\min}^2}{2} \Rightarrow \frac{dx}{dt} = v = \sqrt{\frac{2U_0}{m}}\left(1-\frac{x^2}{a^2}\right)$$

We note that, in the ideal case, because the initial velocity is the escape velocity found in d), ideally x is always smaller or equal to a, then from the above expression,

$$t = \sqrt{\frac{m}{2U_0}}\int_0^x \frac{dx}{\left(1-\frac{x^2}{a^2}\right)} = \sqrt{\frac{ma^2}{8U_0}}\ln\frac{a+x}{a-x} \Rightarrow x(t) = \frac{a\left(\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right)-1\right)}{\left(\exp\left(t\sqrt{\frac{8U_0}{ma^2}}\right)+1\right)}$$



3-1.

a) $$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10^4 \text{ dyne/cm}}{10^2 \text{ gram}}} = \frac{10}{2\pi}\sqrt{\frac{\frac{\text{gram}\cdot\text{cm}}{\text{sec}^2\cdot\text{cm}}}{\text{gram}}} = \frac{10}{2\pi}\ \text{sec}^{-1}$$

or,

$$\boxed{\nu_0 \cong 1.6 \text{ Hz}} \tag{1}$$

$$\tau_0 = \frac{1}{\nu_0} = \frac{2\pi}{10}\ \text{sec}$$

or,

$$\boxed{\tau_0 \cong 0.63 \text{ sec}} \tag{2}$$

b) $$E = \frac{1}{2}kA^2 = \frac{1}{2}\times 10^4 \times 3^2 \text{ dyne-cm}$$

so that

$$\boxed{E = 4.5\times 10^4 \text{ erg}} \tag{3}$$

c) The maximum velocity is attained when the total energy of the oscillator is equal to the kinetic energy. Therefore,

$$\frac{1}{2}mv_{\max}^2 = 4.5\times 10^4 \text{ erg}$$

$$\text{v}_{\max} = \sqrt{\frac{2\times 4.5\times 10^4}{100}}$$

or,

$$\boxed{v_{\max} = 30 \text{ cm/sec}} \tag{4}$$

3-10. The amplitude of a damped oscillator is expressed by

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \delta) \tag{1}$$

Since the amplitude decreases to $1/e$ after n periods, we have

$$\beta n T = \beta n \frac{2\pi}{\omega_1} = 1 \tag{2}$$

Substituting this relation into the equation connecting ω_1 and ω_0 (the frequency of undamped oscillations), $\omega_1^2 = \omega_0^2 - \beta^2$, we have

$$\omega_0^2 = \omega_1^2 + \left[\frac{\omega_1}{2\pi n}\right]^2 = \omega_1^2\left[1 + \frac{1}{4\pi^2 n^2}\right] \tag{3}$$

Therefore,

$$\frac{\omega_1}{\omega_0} = \left[1 + \frac{1}{4\pi^2 n^2}\right]^{-1/2} \tag{4}$$

so that

$$\boxed{\frac{\omega_1}{\omega_2} \cong 1 - \frac{1}{8\pi^2 n^2}}$$

3-12.

The equation of motion is

$$-m\ell\ddot{\theta} = mg\sin\theta \tag{1}$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{2}$$

If θ is sufficiently small, we can approximate $\sin\theta \cong \theta$, and (2) becomes

$$\ddot{\theta} = -\frac{g}{\ell}\theta \tag{3}$$

which has the oscillatory solution

$$\theta(t) = \theta_0\cos\omega_0 t \tag{4}$$

where $\omega_0 = \sqrt{g/\ell}$ and where θ_0 is the amplitude. If there is the retarding force $2m\sqrt{g\ell}\,\dot{\theta}$, the equation of motion becomes

$$-m\ell\ddot{\theta} = mg\sin\theta + 2m\sqrt{g\ell}\,\dot{\theta} \tag{5}$$

or setting $\sin\theta \cong \theta$ and rewriting, we have

$$\ddot{\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0 \tag{6}$$

Comparing this equation with the standard equation for damped motion [Eq. (3.35)],

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{7}$$

we identify $\omega_0 = \beta$. This is just the case of *critical damping*, so the solution for $\theta(t)$ is [see Eq. (3.43)]

$$\theta(t) = (A + Bt)e^{-\omega_0 t} \tag{8}$$

For the initial conditions $\theta(0) = \theta_0$ and $\theta(0) = 0$, we find

$$\boxed{\theta(t) = \theta_0(1 + \omega_0 t)e^{-\omega_0 t}}$$

3-22. For overdamped motion, the position is given by Equation (3.44)

$$x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t} \tag{1}$$

The time derivative of the above equation is, of course, the velocity:

$$v(t) = -A_1\beta_1 e^{-\beta_1 t} - A_2\beta_2 e^{-\beta_2 t} \quad (2)$$

a) At $t = 0$:

$$x_0 = A_1 + A_2 \quad (3)$$

$$v_0 = -A_1\beta_1 - A_2\beta_2 \quad (4)$$

The initial conditions x_0 and v_0 can now be used to solve for the integration constants A_1 and A_2.

b) When $A_1 = 0$, we have $v_0 = -\beta_2 x_0$ and $v(t) = -\beta_2 x(t)$ quite easily. For $A_1 \neq 0$, however, we have $v(t) \to -\beta_1 A_1 e^{-\beta_1 t} = -\beta_1 x$ as $t \to \infty$ since $\beta_1 < \beta_2$.