PHY 401 Fall 2004. Solutions to HW3.

**2-52.**

a) \( F(x) = -\frac{dU}{dx} = -\frac{4U_0 x}{a^2} \left(1 - \frac{x^2}{a^2}\right) \)

b) 

![Graph of potential energy function](image)

When \( F = 0 \), there is equilibrium; further when \( U \) has a local minimum (i.e. \( dF/dx < 0 \)) it is stable, and when \( U \) has a local maximum (i.e. \( dF/dx > 0 \)) it is unstable.

So one can see that in this problem \( x = a \) and \( x = -a \) are unstable equilibrium positions, and \( x = 0 \) is a stable equilibrium position.

c) Around the origin, \( F = -\frac{4U_0 x}{a^2} \equiv -kx \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4U_0}{ma^2}} \)

d) To escape to infinity from \( x = 0 \), the particle needs to get at least to the peak of the potential,

\[
\frac{mv_{\text{esc}}^2}{2} = U_{\text{max}} = U_0 \Rightarrow v_{\text{esc}} = \sqrt{\frac{2U_0}{m}}
\]

e) From energy conservation, we have

\[
\frac{mv^2}{2} + \frac{U_0 x^2}{a^2} = \frac{mv_{\text{esc}}^2}{2} \Rightarrow \frac{dx}{dt} = v = \sqrt{\frac{2U_0}{m}} \left(1 - \frac{x^2}{a^2}\right)
\]

We note that, in the ideal case, because the initial velocity is the escape velocity found in d), ideally \( v \) is always smaller or equal to \( a \), then from the above expression,

\[
t = \int\sqrt{\frac{m}{2U_0}} \left(\frac{dx}{\sqrt{1 - \frac{x^2}{a^2}}}\right) = \sqrt{\frac{ma^2}{8U_0}} \ln \frac{a + \sqrt{a^2 - x^2}}{x} \Rightarrow x(t) = \frac{a}{2} \left(\exp\left(i \sqrt{\frac{8U_0}{ma^2}}\right)ight) - 1 \right) 
\]

\[
\exp\left(i \sqrt{\frac{8U_0}{ma^2}} + 1\right)
\]

![Graph of position over time](image)
3-1.

a) \[ v_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} - \frac{1}{2\pi} \sqrt{\frac{10^8 \text{ dynes/cm}}{10^2 \text{ gram}}} - \frac{10}{2\pi} \sqrt{\frac{\text{ gram} \cdot \text{cm}}{\text{sec}^2 \cdot \text{cm}}} - \frac{10}{2\pi} \text{sec}^{-3} \]
or,
\[ v_0 \equiv 1.6 \text{ Hz} \]  
\[ \tau_0 = \frac{1}{v_0} = \frac{2\pi}{10} \text{ sec} \]
or,
\[ \tau_0 \equiv 0.63 \text{ sec} \]  

b) \[ E = \frac{1}{2} kA^2 = \frac{1}{2} \times 10^8 \times 3^2 \text{ dynes-cm} \]
so that
\[ E = 4.5 \times 10^4 \text{ erg} \]  

3-10. The amplitude of a damped oscillator is expressed by
\[ x(t) = Ae^{-\beta t} \cos(\omega_1 t + \delta) \]  
(1)
Since the amplitude decreases to 1/e after n periods, we have
\[ \beta n T = \beta \frac{2\pi}{\omega_1} = 1 \]  
(2)
Substituting this relation into the equation connecting \( \omega_1 \) and \( \omega_0 \) (the frequency of undamped oscillations), \( \omega_1^2 = \omega_0^2 - \beta^2 \), we have
\[ \omega_0^2 = \omega_1^2 + \left[ \frac{\omega_1}{2\pi n} \right]^2 = \omega_1^2 \left[ 1 + \frac{1}{4\pi^2 n^2} \right] \]  
(3)
Therefore,
\[ \frac{\omega_1}{\omega_0} = \left[ 1 + \frac{1}{4\pi^2 n^2} \right]^{-\beta^2} \]  
(4)
so that
\[ \frac{\omega_1}{\omega_0} \equiv 1 - \frac{1}{8\pi^2 n^2} \]
3-12.

The equation of motion is

\[ -mt \ddot{\theta} = mg \sin \theta \]  \hspace{1cm} (1)

\[ \ddot{\theta} = \frac{g}{l} \sin \theta \]  \hspace{1cm} (2)

If \( \theta \) is sufficiently small, we can approximate \( \sin \theta \approx \theta \), and (2) becomes

\[ \ddot{\theta} = \frac{g}{l} \theta \]  \hspace{1cm} (3)

which has the oscillatory solution

\[ \theta(t) = \theta_0 \cos \omega_0 t \]  \hspace{1cm} (4)

where \( \omega_0 = \sqrt{g/l} \) and where \( \theta_0 \) is the amplitude. If there is the retarding force \( 2m \sqrt{g/l} \dot{\theta} \), the equation of motion becomes

\[ -mt \ddot{\theta} = mg \sin \theta + 2m \sqrt{g/l} \dot{\theta} \]  \hspace{1cm} (5)

or setting \( \sin \theta \approx \theta \) and rewriting, we have

\[ \ddot{\theta} + 2\omega_0 \dot{\theta} + \omega_0^2 \theta = 0 \]  \hspace{1cm} (6)

Comparing this equation with the standard equation for damped motion [Eq. (3.35)],

\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0 \]  \hspace{1cm} (7)

we identify \( \omega_0 = \beta \). This is just the case of critical damping, so the solution for \( \theta(t) \) is [see Eq. (3.43)]

\[ \theta(t) = (A + Bt)e^{-\alpha t} \]  \hspace{1cm} (8)

For the initial conditions \( \theta(0) = \theta_0 \) and \( \dot{\theta}(0) = 0 \), we find

\[ \theta(t) = \theta_0 (1 + \omega_0 t)e^{-\omega_0 t} \]

3-22. For overdamped motion, the position is given by Equation (3.44)

\[ x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t} \]  \hspace{1cm} (1)
The time derivative of the above equation is, of course, the velocity:

$$v(t) = -A_1 \beta_1 e^{-\beta_1 t} - A_2 \beta_2 e^{-\beta_2 t}$$  \hspace{1cm} (2)

**a)** At $t = 0$:

$$x_0 = A_1 + A_2 \hspace{1cm} (3)$$

$$v_0 = -A_1 \beta_1 - A_2 \beta_2 \hspace{1cm} (4)$$

The initial conditions $x_0$ and $v_0$ can now be used to solve for the integration constants $A_1$ and $A_2$.

**b)** When $A_1 = 0$, we have $v_0 = -\beta_2 x_0$ and $v(t) = -\beta_2 x(t)$ quite easily. For $A_1 \neq 0$, however, we have $v(t) \to -\beta_1 A_1 e^{-\beta_1 t} = -\beta_1 x$ as $t \to \infty$ since $\beta_1 < \beta_2$. 