Lecture 4
Superspace
Outline

- Geometric realization of supersymmetry: superspace
- General scalar superfields.
- Examples: chiral superfield, vector superfield, the (spinorial) field strength superfield.
- Extended gauge invariance, and the WZ-gauge.
- Lagrangians in superspace.

Superspace

The idea: generalize the geometric interpretation of the Lorentz group, as the symmetry of 4D Minkowski space.

Implementation: supersymmetry is the symmetry of superspace, an extension of 4D Minkowski space to include additional “fermionic” directions.

Superspace coordinates: \((x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})\) where \(\theta_\alpha, \bar{\theta}_{\dot{\alpha}} = \theta_\alpha^\dagger\) are anticommuting (Grassmann) two-component spinors satisfying

\[
\{\theta_\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0
\]

Supersymmetry is the symmetry group of \((x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})\). The Lorentz subgroup acts according to the Lorentz indices, but other SUSY generators mix bosonic and fermionic coordinates.
Integrals over Superspace

Reminder: for a single component Grassmann variable $\eta$, integrals are defined so:

$$\int d\eta = 0 \ , \ \int d\eta \ \eta = 1$$

For chiral superspace coordinates define:

$$d^2 \theta \equiv -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha \beta}$$

Then

$$\int d^2 \theta \ \theta^2 = -\frac{1}{4} \int d\theta^\alpha d\theta^\beta \epsilon_{\alpha \beta}(\theta^\sigma \epsilon_{\sigma \tau} \theta^\tau)$$

$$= -\frac{1}{4} \epsilon_{\alpha \beta} \epsilon_{\sigma \tau}(\delta^\beta_\sigma \delta^\tau_\alpha - \delta^\alpha_\sigma \delta^\tau_\beta)$$

$$= -\frac{1}{2} \epsilon_{\alpha \beta} \epsilon_{\beta \alpha} = 1$$

Similarly, for arbitrary spinors $\chi$ and $\psi$:

$$\int d^2 \theta (\chi \theta)(\psi \theta) = -\frac{1}{2}(\chi \psi)$$
The analogous superspace measure for the conjugate superspace coordinate:

\[ d^2 \bar{\theta} \equiv -\frac{1}{4} d\bar{\theta}_\alpha d\bar{\theta}_{\bar{\beta}} \epsilon^{\dot{\alpha} \dot{\beta}} \]

Notation for the nonchiral measure over all of the fermionic coordinates

\[ d^4 \theta \equiv d^2 \theta d^2 \bar{\theta} \]

The measure over all of superspace

\[ d^4 x \ d^4 \theta \]

It is this measure that is invariant under all SUSY transformations.
Component expansion of scalar field on superspace (warning: notation does not agree in detail with ours):

\[ \Phi(x, \theta, \bar{\theta}) = f(x) + \theta \phi(x) + \bar{\theta} \bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) + \theta \sigma^\mu \bar{\theta} v_\mu(x) + \theta^2 \bar{\theta} \bar{\lambda}(x) + \bar{\theta}^2 \theta \psi(x) + \theta^2 \bar{\theta}^2 \bar{d}(x) \]

The component fields \( f(x), m(x), n(x), v_\mu(x), d(x) \) are complex bosonic fields.

The component fields \( \phi(x), \bar{\chi}(x), \bar{\lambda}(x), \psi(x) \) are fermionic fields that anti-commute with each other and the superspace coordinates \( \theta, \bar{\theta} \).

There are \( 2 + 2 + 2 + 8 + 2 = 16 \) bosonic components and \( 4 + 4 + 4 + 4 = 16 \) fermionic components.
Superspace transformations

Goal: realize the superalgebra on superspace.

Translate (left-multiply) by $G(y, \epsilon, \bar{\epsilon})$ on the coset element $\Omega(x, \theta, \bar{\theta})$ (="the wave function"): 

$$G(y, \epsilon, \bar{\epsilon})\Omega(x, \theta, \bar{\theta}) = e^{i(-y^\mu P_\mu + \epsilon Q + \bar{\epsilon} \bar{Q})} e^{i(-x^\mu P_\mu + \theta Q + \bar{\theta} \bar{Q})}$$

$$= e^{i(-(x^\mu + y^\mu) P_\mu + (\theta + \epsilon) Q + (\bar{\theta} + \bar{\epsilon}) \bar{Q} + \frac{i}{2} [\epsilon Q, \bar{\theta} \bar{Q}] + \frac{i}{2} [\epsilon \bar{Q}, \theta Q])}$$

$$= \Omega(x^\mu + y^\mu - i\epsilon \sigma^\mu \bar{\theta} + i\theta \sigma^\mu \bar{\epsilon}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon})$$

Intermediate steps

- The Baker-Campbell-Hausdorff formula (the . . . -terms vanish presently):

  $$e^A e^B = e^{A+B + \frac{1}{2} [A,B] + ...}$$

- The fundamental anticommutator in the form

  $$[\epsilon Q, \bar{\theta} \bar{Q}] = 2\epsilon \sigma^\mu \bar{\theta} P_\mu$$
The result: the translation generator and the supertranslation generators are realized on superspace as

\[
\begin{align*}
  i\partial_\mu &= P_\mu \\
  Q_\alpha &= -i \left( \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \right) \\
  Q^\dagger_{\dot{\alpha}} &= i \left( \bar{\partial}_{\dot{\alpha}} + i\theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu \right)
\end{align*}
\]

where (as usual) the spinorial derivatives:

\[
\begin{align*}
  \partial_\alpha &= \frac{\partial}{\partial \theta^\alpha} \\
  \bar{\partial}_{\dot{\alpha}} &= \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}
\end{align*}
\]

act on the left.

Examples:

\[
\begin{align*}
  \partial_\alpha \theta^2 &= \partial_\alpha (\theta^\beta \theta_\beta) = 2\theta_\alpha \\
  \bar{\partial}_{\dot{\alpha}} \bar{\theta}^2 &= \bar{\partial}_{\dot{\alpha}} (\bar{\theta}^{\dot{\beta}} \bar{\theta}_{\dot{\beta}}) = -2\theta_{\dot{\alpha}}
\end{align*}
\]
SUSY transformation of superfield

The SUSY transform of a general scalar superfield:

\[ \delta_\epsilon \Phi(x, \theta, \bar{\theta}) = i(\epsilon Q + \epsilon^\dagger Q^\dagger)\Phi(x, \theta, \bar{\theta}) \]

\[ = \epsilon \phi + \bar{\epsilon} \bar{\chi} + i \theta \sigma^\mu \bar{\epsilon} \partial_\mu f + 2 \epsilon \theta m + \theta \sigma^\mu \bar{\epsilon} v_\mu - i \epsilon \sigma^\mu \bar{\theta} \partial_\mu f \]

\[ + 2 \bar{\epsilon} \theta n + \epsilon \sigma^\mu \bar{\theta} v_\mu + i(\theta \sigma^\mu \epsilon) \partial_\mu \phi + \theta^2 \bar{\epsilon} \lambda - i(\epsilon \sigma^\mu \bar{\theta} \theta \sigma_\mu \bar{\chi}) \]

\[ + \bar{\theta}^2 \epsilon \psi - i \epsilon \sigma^\mu \bar{\theta} \partial_\mu \phi + \epsilon \sigma^\mu \bar{\epsilon} \theta \partial_\mu \bar{\chi} + 2 \epsilon \theta \bar{\theta} \lambda + 2 \bar{\epsilon} \theta \theta \psi \]

\[ - i \epsilon \sigma^\mu \bar{\theta} \theta^2 \partial_\mu m + i \theta \sigma^\mu \bar{\epsilon} \theta \sigma_\nu \bar{\theta} \partial_\mu v_\nu + 2 \theta^2 \bar{\epsilon} \bar{\theta} d + i \theta \sigma^\mu \bar{\epsilon} \theta^2 \partial_\mu n \]

\[ - i \epsilon \sigma^\mu \bar{\theta} \theta \sigma_\nu \bar{\theta} \partial_\mu v_\nu + 2 \epsilon \theta \bar{\theta}^2 d - i \epsilon \sigma^\mu \bar{\theta} \theta^2 \partial_\mu \lambda + i \theta \sigma^\mu \bar{\epsilon} \theta^2 \theta \partial_\mu \psi \]

The details are evidently tedious to verify (and details do not conform with our notation). But the general form is clear with a minimum of computation.
SUSY transformation of components

Reorganize expression into standard expansion of the scalar field in components. ⇒ the SUSY transformations in component form:

\[
\begin{align*}
\delta_{\epsilon} f &= \epsilon \phi + \bar{\epsilon} \bar{\chi} \\
\delta_{\epsilon} \phi_{\alpha} &= 2 \epsilon_{\alpha} m + \sigma_{\alpha \beta}^{\mu} \bar{\epsilon} \beta (i \partial_{\mu} f + v_{\mu}) \\
\delta_{\epsilon} \bar{\chi}^{\dot{\alpha}} &= 2 \bar{\epsilon}^{\dot{\alpha}} n + \epsilon^{\beta} \sigma^{\mu}_{\beta \dot{\gamma}} \bar{\epsilon} \dot{\gamma} \dot{\alpha} [i \partial_{\mu} f - v_{\mu}] \\
\delta_{\epsilon} m &= \bar{\epsilon} \bar{\lambda} - \frac{i}{2} \partial_{\mu} \phi \sigma^{\mu} \bar{\epsilon} \\
\delta_{\epsilon} n &= \epsilon \psi + \frac{i}{2} \epsilon \sigma^{\mu} \partial_{\mu} \bar{\lambda} \\
\delta_{\epsilon} v_{\mu} &= \epsilon \sigma^{\mu} \bar{\lambda} + \psi \sigma^{\mu} \bar{\epsilon} + \frac{i}{2} \epsilon \partial_{\mu} \phi - \frac{i}{2} \partial_{\mu} \bar{\chi} \epsilon \\
\delta_{\epsilon} \bar{\lambda}^{\dot{\alpha}} &= 2 \bar{\epsilon}^{\dot{\alpha}} d + \frac{i}{2} \bar{\sigma}^{\dot{\alpha}} \partial_{\mu} v_{\mu} + i (\epsilon \sigma^{\mu} \bar{e}) \dot{\alpha} \partial_{\mu} m \\
\delta_{\epsilon} \psi_{\alpha} &= 2 \epsilon_{\alpha} d - \frac{i}{2} \epsilon_{\alpha} \partial_{\mu} v_{\mu} + i (\sigma^{\mu} \bar{e})_{\alpha} \partial_{\mu} n \\
\partial_{\epsilon} d &= \frac{i}{2} \partial_{\mu} [\psi \sigma^{\mu} \bar{e} + \epsilon \sigma^{\mu} \bar{\lambda}] \\
\end{align*}
\]

Important result: the highest component of the superfield (the $d$-field) transforms to a total derivative ($\theta^2 \bar{\theta}^2$ cannot result after $\partial_{\alpha}, \partial_{\dot{\alpha}}$ derivative, only after $\partial_{\mu}$ derivative.)
Constraints

The general superfield represents the SUSY algebra, but the representation is reducible.

Example: the constraints

\[
\begin{align*}
\chi(x) &= 0 \\
n(x) &= 0 \\
v_\mu(x) &= i\partial_\mu f(x) \\
\bar{\lambda} &= \frac{i}{2} \partial_\mu \phi(x) \sigma^\mu \\
\psi(x) &= 0 \\
d(x) &= -\frac{i}{4} \Box f(x)
\end{align*}
\]

are preserved by the SUSY transform.

Remark: the irreducible representation defined by these constraints is the chiral supermultiplet (in an unfamiliar notation).
The superspace derivatives

\[ i \partial_\mu = P_\mu \]
\[ D_\alpha = \partial_\alpha - i \sigma^{\mu}_{\alpha \beta} \bar{\theta}^\beta \partial_\mu \]
\[ \bar{D}_{\dot{\alpha}} = - \left( \bar{\partial}_{\dot{\alpha}} - i \theta^{\beta} \sigma^{\mu}_{\beta \dot{\alpha}} \partial_\mu \right) \]

are designed such that they anti-commute with \( Q_\alpha, Q_\dot{\alpha}^\dagger \).

They satisfy the fundamental anticommutator with the “wrong” sign

\[ \{ D_\alpha, \bar{D}_{\dot{\alpha}} \} = -2 \sigma^{\mu}_{\alpha \dot{\alpha}} P_\mu \]

Example: if the components of a general superfields satisfy

\[ \bar{D}_{\dot{\alpha}} \Phi = 0 \]

then the transformed field will also satisfy this constraint.

This defines the chiral superfield in superspace notation.
Constrained Superfields

Define the superspace coordinate $y^\mu$ such that $\bar{D}_\alpha y^\mu = 0$:

$$y^\mu = x^\mu - i\theta \sigma^\mu \bar{\theta}$$

Then the general solution to $\bar{D}_\alpha \Phi = 0$ is a superfield of the form $\Phi(y, \theta)$.

Expand the chiral superfield in components (recalling that squares of Grassmann variables vanish):

$$\Phi(y, \theta) \equiv \phi(y) + \sqrt{2} \theta \psi(y) + \theta^2 \mathcal{F}(y)$$

$$= \phi(x) - i\theta \sigma^\mu \bar{\theta} \partial_\mu \phi(x) - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi(x)$$

$$+ \sqrt{2} \theta \psi(x) + \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi(x) \sigma^\mu \bar{\theta} + \theta^2 \mathcal{F}(x)$$

Remark: the SUSY variation of the highest component (the $\mathcal{F}$-term) must be a total derivative.
The kinetic terms in superspace

The kinetic term of a chiral field in superspace formalism:

\[
\int d^4\theta \Phi^\dagger \Phi = \int d^4\theta \left( \phi^* + i\theta\sigma^\mu \bar{\theta} \partial_\mu \phi^* - \frac{1}{4} \bar{\theta}^2 \theta^2 \partial^2 \phi^* 
+ \sqrt{2}\bar{\theta}\psi^\dagger - \frac{i}{\sqrt{2}} \theta\sigma^\mu \partial_\mu \psi^\dagger \bar{\theta}^2 + \bar{\theta}^2 \mathcal{F}^* \right) 
\cdot \left( \phi - i\theta\sigma^\mu \theta \partial_\mu \phi - \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi 
+ \sqrt{2}\theta\psi + \frac{i}{\sqrt{2}} \theta^2 \partial_\mu \psi \sigma^\mu \bar{\theta} + \theta^2 \mathcal{F} \right)
\]

\[
= \mathcal{F}^* \mathcal{F} + \partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi 
- \frac{1}{4} \partial^\mu (\phi^* \partial_\mu \phi + \partial_\mu \phi^* \phi) - \frac{i}{2} \partial_\mu (\psi^\dagger \bar{\sigma}^\mu \psi)
\]

Identities:

- \( \psi^\dagger \bar{\sigma}^\mu \chi = -\chi\sigma^\mu \psi^\dagger \)
- \( \theta\sigma^\mu \bar{\theta} \theta\sigma^\nu \bar{\theta} = \frac{1}{2} \eta^{\mu\nu} \theta^2 \bar{\theta}^2 \)

Summary: the superspace integral over the kinetic term of a chiral field:

\[
\int d^4x d^4\theta \Phi^\dagger \Phi = \int d^4x \left( \mathcal{F}^* \mathcal{F} + \partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi \right)
= \int d^4x \mathcal{L}_{\text{free}}
\]
Superpotential

The derivative $\bar{D}_\alpha$ satisfies the chain rule so the product of chiral superfields is also a chiral superfield.

The superpotential $W(\Phi)$ is a power series in the chiral superfield, so it is itself a chiral superfield.

The SUSY variation of the $\theta^2$ component of a chiral superfield is a total derivative so we can always form the SUSY invariant

$$\int d^4x d^2\theta W(\Phi) = \int d^4x W|_{\theta^2\text{--only}}(\Phi)$$

Component expansion (see next slide for details):

$$\int d^2\theta W(\Phi) = W_a \mathcal{F}^a - \frac{1}{2} W_{ab} \psi^a \psi^b - \partial_\mu \left( \frac{1}{4} W^a \bar{\theta}^2 \partial^\mu \phi_a - \frac{i}{\sqrt{2}} W^a \psi_a \sigma^\mu \bar{\theta} \right)$$

The interaction terms of the WZ-model in superspace notation:

$$\int d^4x d^2\theta W(\Phi) + h.c. = \int d^4x \mathcal{L}_{\text{int}}$$
Details

To work out the component expansion, first expand $y$:

$$W = W_{y=x} + \partial_\mu W|_{y=x} (-i\theta\sigma^\mu \bar{\theta}) + \frac{1}{2} \partial_\mu \partial_\nu W|_{y=x} (-i\theta\sigma^\mu \bar{\theta})(-i\theta\sigma^\nu \bar{\theta})$$

Up to total derivatives we can thus take $y = x$ and immediately find

$$W|_{\theta^2-\text{only}} = \left[ \partial_a W F^a \theta^2 + \frac{1}{2} \partial_a \partial_b W (\sqrt{2}\theta \psi^a)(\sqrt{2}\theta \psi^b) \right]_{\theta^2-\text{only}}$$

$$= \partial_a W F^a - \frac{1}{2} \partial_a \partial_b W \psi^a \theta \psi^b$$

Expanding the derivative terms in $\theta$ we find their component form:

$$\partial_\mu \left[ \partial_a W \sqrt{2}\theta \psi^a (-i\theta\sigma^\mu \bar{\theta}) - \frac{1}{4} \partial^\mu W \theta^2 \bar{\theta}^2 \right]_{\theta^2-\text{only}}$$

$$= \partial_\mu \left[ \frac{i}{\sqrt{2}} \partial_a W (\psi^a \sigma^\mu \bar{\theta}) - \frac{1}{4} W_a \partial^\mu \phi^a \bar{\theta}^2 \right]$$
Kähler function

More general Lagrangian:

$$\int d^4x \, d^4\theta \, K(\Phi^\dagger, \Phi)$$

SUSY invariance: the SUSY variation of the $\theta^2 \bar{\theta}^2$ component of any superfield is a total derivative.

Terminology: the real function $K$ is called the Kähler function.

It gives general kinetic and other (in general non-renormalizable) interactions.
The Real Superfield

Reality is another constraint that can be consistently imposed on a general scalar superfield:

\[ V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta}) \]

Parametrize the real superfield conveniently

\[ V(x, \theta, \bar{\theta}) = C(x) + \theta \chi(x) + \bar{\theta} \chi^\dagger(x) + \frac{1}{2} \theta^2 [M(x) + iN(x)] \]
\[ + \frac{1}{2} \bar{\theta}^2 [M(x) - iN(x)] + \theta \sigma^\mu \partial_\mu \chi(x) + \theta^2 [\lambda^\dagger(x) + \frac{1}{2} \partial_\mu \chi(x) \sigma^\mu] \bar{\theta} \]
\[ + \theta^2 \partial^2 [D(x) - \frac{1}{2} \Box C(x)] \]

Example: real superfield constructed from a chiral superfield \( \Lambda \)

\[ \Lambda + \Lambda^\dagger = (A + A^*) + \sqrt{2} \theta \psi + \sqrt{2} \bar{\theta} \psi^\dagger + \theta^2 F + \bar{\theta}^2 F^* - i \theta \sigma^\mu \bar{\theta} \partial_\mu (A - A^*) \]
\[ + \frac{1}{\sqrt{2}} \theta^2 \partial_\mu \psi \sigma^\mu \bar{\theta} + \frac{1}{\sqrt{2}} \bar{\theta}^2 \theta \sigma^\mu \partial_\mu \psi^\dagger - \frac{1}{4} \theta^2 \bar{\theta}^2 \Box (A + A^*) \]
Extended gauge invariance

Extended gauge invariance acts on the entire real superfield:

$$V \rightarrow V + \Lambda + \Lambda^\dagger$$

It acts as ordinary gauge invariance on the vector field component $A_\mu$:

$$A_\mu \rightarrow A_\mu - i\partial_\mu(A - A^*)$$

A SUSY theory invariant under ordinary gauge symmetry is automatically invariant under the full extended gauge symmetry, where the gauge parameter is a chiral superfield, $\Lambda$. 
The WZ gauge

The WZ-gauge: choose gauge functions of extended gauge invariance so three scalar fields $C(x), M(x), N(x)$ and the Weyl spinors $\chi(x), \chi^\dagger(x)$ are transformed to zero.

The vector supermultiplet in the Wess-Zumino gauge:

$$V^a_{\text{WZ}} = \theta \sigma^\mu \bar{\theta} A^a_\mu + \theta^2 \bar{\theta} \lambda^a + \bar{\theta}^2 \theta \lambda^a + \frac{1}{2} \theta^2 \bar{\theta}^2 D^a,$$

Remark: adjoint indices appropriate for non-abelian generalization. The transformation to WZ-gauge is (much) more complicated in that case, but the final result is the same.
Non abelian gauge invariance

Extended gauge transformation:

\[ \exp(T^a V^a) \rightarrow \exp(T^a \Lambda^a \dagger) \exp(T^a V^a) \exp(T^a \Lambda^a) \]

Linear approximation:

\[ V^a \rightarrow V^a + \Lambda^a + \Lambda^a \dagger + \mathcal{O}(V^a \Lambda^a) \]

Higher orders simplify in Wess–Zumino gauge:

\[ V^a V^b = \frac{1}{2} \theta^2 \bar{\theta}^2 A^{a \mu} A^b_\mu \]
\[ V^a V^b V^c = 0 \]
“Field strength” Chiral Superfield

Goal: make extended gauge symmetry manifest, and see that $C(x), M(x), C(x)$ and $\chi(x), \chi^\dagger(x)$ are unphysical (they decouple).

Starting from a vector superfield, define the spinorial superfield

\[ T^a W^a_\alpha = -\frac{1}{4} \bar{D}_\dot{\alpha} \bar{D}^{\dot{\alpha}} e^{-T^a V^a} D^a e^{T^a V^a} \]

Expand in components:

\[ W^a_\alpha = \lambda^a_\alpha(y) + \theta^a_\alpha D^a_\alpha(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) + \theta^2 i\sigma^\mu D_\mu \lambda^\dagger_a(y), \]

where

\[ \sigma^{\mu\nu} = \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \]

Terminology: $W_\alpha$ is the “field strength” chiral superfield.

Interpretation: $W_\alpha$ isolates the physical degrees of freedom from the vector superfield (off-shell, the $D$-term is included).
Why is $W_\alpha$ “chiral”? 

Recall: chiral scalar superfields are general superfields satisfying

$$\bar{D}_\dot{\alpha} \Phi = 0$$

Definition of field strength superfield:

$$T^a W^a_\alpha = -\frac{1}{4} \bar{D}_\dot{\alpha} \bar{D}^\dot{\alpha} e^{-T^a V^a} D_\alpha e^{T^a V^a}$$

There are just two distinct super-derivatives $\bar{D}_\dot{\alpha}$ ($\dot{\alpha} = \dot{1}$ and $\dot{\alpha} = \dot{2}$) and they “all” anti-commute; so

$$\bar{D}_\dot{\alpha} W^a_\alpha = 0$$

Conclusion: $W^a_\alpha$ is chiral.

Corollary: the chiral superfield $W^a_\alpha$ is a function of the superspace coordinate $y^\mu = x^\mu - i \theta \sigma^\mu \bar{\theta}$ and $\theta$, but not $\bar{\theta}$.

(Proof: these are the quantities annihilated by $\bar{D}_\dot{\alpha}$.)
Component Expansion: SUSY QED

Goal: work out the field strength superfield in components.

Definition:

\[ W_\alpha = -\frac{1}{4} \bar{D}_\dot{\alpha} \bar{D}^{\dot{\alpha}} (D_\alpha V) = D_\alpha V \big|_{\bar{\theta}^2 \text{ only}} \]

where the vector field (in WZ-gauge):

\[ V_{WZ} = \theta \sigma^\mu \bar{\theta} A_\mu + \theta^2 \bar{\theta} \lambda^\dagger + \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D , \]

and the superspace derivative (with the \( y^\mu \) coordinate kept fixed):

\[ D_\alpha = \frac{\partial}{\partial \theta_\alpha} - 2i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial y^\mu} \]

Manipulations (details below):

\[ W_\alpha = \lambda_\alpha + \theta_\alpha D - 2i(\sigma^{\mu \bar{\theta}})_\alpha [(\theta \sigma^\nu \bar{\theta} \partial_\mu A_\nu + \theta^2 \bar{\theta} \partial_\mu \lambda^\dagger)_{\bar{\theta}^2 \text{ only}} \]

\[ = \lambda_\alpha + \theta_\alpha D - (\sigma^{\mu \nu} \theta)_{\alpha} F_{\mu \nu} + i\theta^2 (\sigma^\mu \partial_\mu \lambda^\dagger \alpha \]
Reordering:

\[(\sigma^\mu \bar{\theta})_{\alpha} \theta \sigma^\nu \bar{\theta} = -\sigma^\mu_{\alpha \dot{\alpha}} \bar{\theta} \dot{\alpha} \bar{\theta} \bar{\nu} \dot{\beta} \beta \theta \beta) = \frac{1}{2} (\sigma^\mu \bar{\sigma}^\nu \theta)_{\alpha} \bar{\theta}^2\]

where

\[\sigma^{\mu \nu} = \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)\]

Some identities:

\[\psi \sigma^\mu \chi^\dagger = -\chi^\dagger \bar{\sigma}^\mu \psi\]

\[\bar{\theta} \dot{\alpha} \theta \dot{\beta} = -\frac{1}{2} \bar{\theta}^2 \delta^\dot{\alpha}_{\dot{\beta}}\]

\[(\psi \theta)(\chi \theta) = -\frac{1}{2} (\psi \chi) \theta^2\]

Each component field depends on \(y^\mu\), the entire \(W_\alpha\) depends on \(y^\mu, \theta\):

\[W_\alpha(y, \theta) = \lambda_\alpha(y) + \theta_\alpha D(y) - (\sigma^{\mu \nu} \theta)_{\alpha} F_{\mu \nu}(y) + i \theta^2 (\sigma^\mu \partial_\mu \lambda^\dagger(y))_{\alpha}\]
SUSY Yang–Mills action

Lagrangian in superspace notation:

$$\int d^4x \mathcal{L}_{SYM} = \frac{1}{4} \int d^4x d^2\theta \ W^a\alpha W^a_{\alpha} + h.c.$$ 

where

$$\mathcal{L}_{SYM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\lambda^\dagger a \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a$$

An alternate form, as an integral over all of superspace:

$$\int d^4x \mathcal{L}_{SYM} = \frac{1}{2} \int d^4x d^4\theta \text{Tr} T^a W^a\alpha e^{-T^a V^a} D_\alpha e^{T^a V^a} + h.c.$$ 

The equivalence:

$$T^a W^a\alpha = -\frac{1}{4} \bar{D}\,^2(e^{-T^a V^a} D_\alpha e^{T^a V^a})$$

$$= [e^{-T^a V^a} D_\alpha e^{T^a V^a}]_{\theta^2 \text{ only}}$$

$$= \int d^2\bar{\theta} [e^{-T^a V^a} D_\alpha e^{T^a V^a}]$$
Matter Couplings

The chiral superfield $\Phi$ transforms under the extended gauge invariance as

$$\Phi \to e^{-gT^a \Lambda^a} \Phi$$

Standard gauge invariant kinetic terms:

$$\int d^4 \theta \, \Phi^\dagger e^{gT^a V^a} \Phi$$

More generally (to include non-renormalizable interactions):

$$\int d^4 \theta \, K(\Phi^\dagger, e^{gT^a V^a} \Phi)$$