Lecture 4
Superspace

Outline

- Geometric realization of supersymmetry: superspace
- General scalar superfields.
- Examples: chiral superfield, vector superfield, the (spinorial) field strength superfield.
- Extended gauge invariance, and the WZ-gauge.
- Lagrangians in superspace.

Reading: Terning 2.7, Lykken's TASI notes (arXiv: 9612114), sec 3.

Superspace

The idea: generalize the geometric interpretation of the Lorentz group, as the symmetry of 4D Minkowski space.

Implementation: supersymmetry is the symmetry of superspace, an extension of 4D Minkowski space to include additional "fermionic" directions.

Superspace coordinates: $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ where θ_α, $\bar{\theta}_{\dot{\alpha}} = \theta^\dagger_\alpha$ are anticommuting (Grassmann) two-component spinors satisfying

$$\{\theta_\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0$$

Supersymmetry is the symmetry group of $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The Lorentz subgroup acts according to the Lorentz indices, but other SUSY generators mix bosonic and fermionic coordinates.

Integrals over Superspace

Reminder: for a single component Grassmann variable η, integrals are defined so:

$$\int d\eta = 0 \ , \qquad \int d\eta \ \eta = 1$$

For chiral superspace coordinates define:

$$d^2\theta \quad \equiv \quad -\tfrac{1}{4} d\theta^\alpha d\theta^\beta \boldsymbol{\epsilon}_{\alpha\beta}$$

Then

$$\begin{aligned} \int d^2\theta \, \theta^2 \quad &= \quad -\tfrac{1}{4} \int d\theta^\alpha d\theta^\beta \boldsymbol{\epsilon}_{\alpha\beta} (\theta^\sigma \boldsymbol{\epsilon}_{\sigma\tau} \theta^\tau) \\ &= \quad -\tfrac{1}{4} \boldsymbol{\epsilon}_{\alpha\beta} \boldsymbol{\epsilon}_{\sigma\tau} (\delta^{\beta\sigma} \delta^{\tau\alpha} - \delta^{\alpha\sigma} \delta^{\tau\beta}) \\ &= \quad -\tfrac{1}{2} \boldsymbol{\epsilon}_{\alpha\beta} \boldsymbol{\epsilon}_{\beta\alpha} = 1 \end{aligned}$$

Similarly, for arbitrary spinors χ and ψ:

$$\int d^2\theta \, (\chi\theta)(\psi\theta) = -\tfrac{1}{2} (\chi\psi)$$

The analogous superspace measure for the conjugate superspace coordinate:

$$d^2\bar{\theta} \quad \equiv \quad -\tfrac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \boldsymbol{\epsilon}^{\dot{\alpha}\dot{\beta}}$$

Notation for the nonchiral measure over all of the fermionic coordinates

$$d^4\theta \quad \equiv \quad d^2\theta d^2\bar{\theta}$$

The measure over all of superspace

$$d^4x \; d^4\theta$$

It is this measure that is invariant under all SUSY transformations.

General Scalar Superfield

Component expansion of scalar field on superspace (warning: notation does not not agree in detail with ours):

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) \\ &\quad + \theta\sigma^\mu\bar{\theta}v_\mu(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x)\end{aligned}$$

The component fields $f(x)$, $m(x)$, $n(x)$, $v_\mu(x)$, $d(x)$ are complex bosonic fields.

The component fields $\phi(x)$, $\bar{\chi}(x)$, $\bar{\lambda}(x)$, $\psi(x)$ are fermionic fields that anti-commute with each other and the superspace coordinates $\theta, \bar{\theta}$.

There are $2+2+2+8+2 = 16$ bosonic components and $4+4+4+4 = 16$ fermionic components.

Superspace transformations

Goal: realize the superalgebra on superspace.

Translate (left-multiply) by $G(y,\epsilon,\bar{\epsilon})$ on the coset element $\Omega(x,\theta,\bar{\theta})$ (="the wave function"):

$$\begin{aligned} G(y,\epsilon,\bar{\epsilon})\Omega(x,\theta,\bar{\theta}) &= e^{i(-y^\mu P_\mu+\epsilon Q+\bar{\epsilon}\bar{Q})}e^{i(-x^\mu P_\mu+\theta Q+\bar{\theta}\bar{Q})} \\ &= e^{i(-(x^\mu+y^\mu)P_\mu+(\theta+\epsilon)Q+(\bar{\epsilon}+\bar{\theta})\bar{Q}+\frac{i}{2}[\epsilon Q,\bar{\theta}\bar{Q}]+\frac{i}{2}[\bar{\epsilon}\bar{Q},\theta Q])} \\ &= \Omega(x^\mu+y^\mu-i\epsilon\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\epsilon},\theta+\epsilon,\bar{\theta}+\bar{\epsilon}) \end{aligned}$$

Intermediate steps

- The Baker-Campbell-Hausdorff formula (the ...-terms vanish presently):

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$$

- The fundamental anticommutator in the form

$$[\epsilon Q,\bar{\theta}\bar{Q}] = 2\epsilon\sigma^\mu\bar{\theta}P_\mu$$

The result: the translation generator and the supertranslation generators are realized on superspace as

$$\begin{aligned} i\partial_\mu &= P_\mu \\ Q_\alpha &= -i\left(\partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\right) \\ Q^\dagger_{\dot\alpha} &= i\left(\bar\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu\right) \end{aligned}$$

where (as usual) the spinorial derivatives:

$$\begin{aligned} \partial_\alpha &= \frac{\partial}{\partial\theta^\alpha} \\ \bar\partial_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \end{aligned}$$

act on the left.

Examples:

$$\begin{aligned} \partial_\alpha\theta^2 &= \partial_\alpha(\theta^\beta\theta_\beta) = 2\theta_\alpha \\ \bar\partial_{\dot\alpha}\bar\theta^2 &= \bar\partial_{\dot\alpha}(\bar\theta_{\dot\beta}\bar\theta^{\dot\beta}) = -2\theta_{\dot\alpha} \end{aligned}$$

SUSY transformation of superfield

The SUSY transform of a general scalar superfield:

$$
\begin{aligned}
\delta_\epsilon \Phi(x,\theta,\bar\theta) &= i(\epsilon Q + \epsilon^\dagger Q^\dagger)\Phi(x,\theta,\bar\theta) \\
&= \epsilon\phi + \bar\epsilon\bar\chi + i\theta\sigma^\mu\bar\epsilon\partial_\mu f + 2\epsilon\theta m + \theta\sigma^\mu\bar\epsilon v_\mu - i\epsilon\sigma^\mu\bar\theta\partial_\mu f \\
&+ 2\bar\epsilon\bar\theta n + \epsilon\sigma^\mu\bar\theta v_\mu + i(\theta\sigma^\mu\epsilon)\theta\partial_\mu\phi + \theta^2\bar\epsilon\bar\lambda - i(\epsilon\sigma^\mu\bar\theta)\bar\theta\sigma_\mu\bar\chi \\
&+ \bar\theta^2\epsilon\psi - i\epsilon\sigma^\mu\bar\theta\theta\partial_\mu\phi + i\theta\sigma^\mu\bar\epsilon\bar\theta\partial_\mu\bar\chi + 2\epsilon\theta\bar\theta\bar\lambda + 2\bar\epsilon\bar\theta\theta\psi \\
&- i\epsilon\sigma^\mu\bar\theta\theta^2\partial_\mu m + i\theta\sigma^\mu\bar\epsilon\theta\sigma^\nu\bar\theta\partial_\mu v_\nu + 2\theta^2\bar\epsilon\bar\theta d + i\theta\sigma^\mu\bar\epsilon\bar\theta^2\partial_\mu n \\
&- i\epsilon\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta\partial_\mu v_\nu + 2\epsilon\theta\bar\theta^2 d - i\epsilon\sigma^\mu\bar\theta\theta^2\bar\theta\partial_\mu\bar\lambda + i\theta\sigma^\mu\bar\epsilon\bar\theta^2\theta\partial_\mu\psi
\end{aligned}
$$

The details are evidently tedious to verify (and details do not conform with our notation). But the general form is clear with a minimum of computation.

SUSY transformation of components

Reorganize expression into standard expansion of the scalar field in components. $\Rightarrow$ the SUSY transformations in component form:

$$
\begin{aligned}
\delta_\epsilon f &= \epsilon\phi + \bar{\epsilon}\bar{\chi} \\
\delta_\epsilon \phi_\alpha &= 2\epsilon_\alpha m + \sigma^\mu_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}(i\partial_\mu f + v_\mu) \\
\delta_\epsilon \bar{\chi}^{\dot{\alpha}} &= 2\bar{\epsilon}^{\dot{\alpha}} n + \epsilon^\beta \sigma^\mu_{\beta\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}[i\partial_\mu f - v_\mu] \\
\delta_\epsilon m &= \bar{\epsilon}\bar{\lambda} - \tfrac{i}{2}\partial_\mu\phi\sigma^\mu\bar{\epsilon} \\
\delta_\epsilon n &= \epsilon\psi + \tfrac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\
\delta_\epsilon v_\mu &= \epsilon\sigma^\mu\bar{\lambda} + \psi\sigma^\mu\bar{\epsilon} + \tfrac{i}{2}\epsilon\partial_\mu\phi - \tfrac{i}{2}\partial_\mu\bar{\chi}\epsilon \\
\delta_\epsilon \bar{\lambda}^{\dot{\alpha}} &= 2\bar{\epsilon}^{\dot{\alpha}} d + \tfrac{i}{2}\bar{\xi}^{\dot{\alpha}}\partial^\mu v_\mu + i(\epsilon\sigma^\mu\epsilon)^{\dot{\alpha}}\partial_\mu m \\
\delta_\epsilon \psi_\alpha &= 2\epsilon_\alpha d - \tfrac{i}{2}\epsilon_\alpha\partial^\mu v_\mu + i(\sigma^\mu\bar{\epsilon})_\alpha\partial_\mu n \\
\partial_\epsilon d &= \tfrac{i}{2}\partial_\mu[\psi\sigma^\mu\bar{\epsilon} + \epsilon\sigma^\mu\bar{\lambda}]
\end{aligned}
$$

Important result: the highest component of the superfield (the d-field) transforms to a total derivative ($\theta^2\bar{\theta}^2$ cannot result after ∂_α, $\partial_{\dot{\alpha}}$ derivative, only after ∂_μ derivative.)

Constraints

The general superfield represents the SUSY algebra, but the representation is reducible.

Example: the constraints

$$
\begin{aligned}
\chi(x) &= 0 \\
n(x) &= 0 \\
v_\mu(x) &= i\partial_\mu f(x) \\
\bar{\lambda} &= \tfrac{i}{2}\partial_\mu \phi(x)\sigma^\mu \\
\psi(x) &= 0 \\
d(x) &= -\tfrac{i}{4}\Box f(x)
\end{aligned}
$$

are preserved by the SUSY transform.

Remark: the irreducible representation defined by these constraints is the chiral supermultiplet (in an unfamiliar notation).

The Superspace Derivative

The superspace derivatives

$$\begin{aligned} i\partial_\mu &= P_\mu \\ D_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar D_{\dot\alpha} &= -\left(\bar\partial_{\dot\alpha} - i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu\right) \end{aligned}$$

are designed such that they anti-commute with $Q_\alpha, Q^\dagger_{\dot\alpha}$.

They satisfy the fundamental anticommutator with the "wrong" sign

$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

Example: if the components of a general superfields satisfy

$$\bar D_{\dot\alpha}\Phi = 0$$

then the transformed field will also satisfy this constraint.

This defines the chiral superfield in superspace notation.

Constrained Superfields

Define the superspace coordinate y^μ such that $\bar{D}_{\dot{\alpha}} y^\mu = 0$:

$$y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$$

Then the general solution to $\bar{D}_{\dot{\alpha}}\Phi = 0$ is a superfield of the form $\Phi(y, \theta)$.

Expand the chiral superfield in components (recalling that squares of Grassmann variables vanish):

$$\begin{aligned}
\Phi(y,\theta) &\equiv \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2\mathcal{F}(y) \\
&= \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \tfrac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) \\
&\quad + \sqrt{2}\theta\psi(x) + \tfrac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2\mathcal{F}(x)
\end{aligned}$$

Remark: the SUSY variation of the highest component (the $\mathcal{F}$-term) must be a total derivative.

The kinetic terms in superspace

The kinetic term of a chiral field in superspace formalism:

$$\begin{aligned}
\int d^4\theta\, \Phi^\dagger \Phi &= \int d^4\theta \begin{pmatrix} \phi^* + i\theta\sigma^\mu\bar\theta\partial_\mu\phi^* - \frac{1}{4}\bar\theta^2\theta^2\partial^2\phi^* \\ +\sqrt{2}\bar\theta\psi^\dagger - \frac{i}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\psi^\dagger\bar\theta^2 + \bar\theta^2\mathcal{F}^* \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} \phi - i\theta\sigma^\mu\bar\theta\partial_\mu\phi - \frac{1}{4}\theta^2\bar\theta^2\partial^2\phi \\ +\sqrt{2}\theta\psi + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar\theta + \theta^2\mathcal{F} \end{pmatrix} \\
&= \mathcal{F}^*\mathcal{F} + \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi \\
&\quad -\frac{1}{4}\partial^\mu(\phi^*\partial_\mu\phi + \partial_\mu\phi^*\phi) - \frac{i}{2}\partial_\mu(\psi^\dagger\overline{\sigma}^\mu\psi)
\end{aligned}$$

Identities:

- $\psi^\dagger\bar\sigma^\mu\chi = -\chi\sigma^\mu\psi^\dagger$
- $\theta\sigma^\mu\bar\theta\ \theta\sigma^\nu\bar\theta = \frac{1}{2}\eta^{\mu\nu}\theta^2\bar\theta^2$

Summary: the superspace integral over the kinetic term of a chiral field:

$$\begin{aligned}
\int d^4x\, d^4\theta\, \Phi^\dagger\Phi &= \int d^4x\, \left(\mathcal{F}^*\mathcal{F} + \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi\right) \\
&= \int d^4x\, \mathcal{L}_{\text{free}}
\end{aligned}$$

Superpotential

The derivative $\bar{D}_{\dot{\alpha}}$ satisfies the chain rule so the product of chiral superfields is also a chiral superfield.

The superpotential $W(\Phi)$ is a power series in the chiral superfield, so it is itself a chiral superfield

The SUSY variation of the θ^2 component of a chiral superfield is a total derivative so we can always form the SUSY invariant

$$\int d^4x\, d^2\theta\; W(\Phi) = \int d^4x\; W|_{\theta^2-\text{only}}\,(\Phi)$$

Component expansion (see next slide for details):

$$\int d^2\theta\, W(\Phi) \quad = \quad W_a\mathcal{F}^a - \tfrac{1}{2}W_{ab}\psi^a\psi^b - \partial_\mu(\tfrac{1}{4}W^a\bar{\theta}^2\partial^\mu\phi_a - \tfrac{i}{\sqrt{2}}W^a\psi_a\sigma^\mu\bar{\theta})$$

The interaction terms of the WZ-model in superspace notation:

$$\int d^4x\, d^2\theta\, W(\Phi) + h.c. = \int d^4x\, \mathcal{L}_{\text{int}}$$

Details

To work out the component expansion, first expand y:

$$W \quad = W_{y=x} + \partial_\mu W|_{y=x}(-i\theta\sigma^\mu\bar{\theta}) + \tfrac{1}{2}\,\partial_\mu\partial_\nu W|_{y=x}(-i\theta\sigma^\mu\bar{\theta})(-i\theta\sigma^\nu\bar{\theta})$$

Up to total derivatives we can thus take $y = x$ and immediately find

$$\begin{aligned} W|_{\theta^2-\text{only}} &= \left[\partial_a W\mathcal{F}^a\theta^2 + \tfrac{1}{2}\partial_a\partial_b W(\sqrt{2}\theta\psi^a)(\sqrt{2}\theta\psi^b)\right]_{\theta^2-\text{only}} \\ &= \partial_a W\mathcal{F}^a - \tfrac{1}{2}\partial_a\partial_b W\psi^a\theta\psi^b \end{aligned}$$

Expanding the derivative terms in θ we find their component form:

$$\begin{aligned} & \partial_\mu\left[\partial_a W\sqrt{2}\theta\psi^a(-i\theta\sigma^\mu\bar{\theta}) - \tfrac{1}{4}\partial^\mu W\theta^2\bar{\theta}^2\right]_{\theta^2-\text{only}} \\ = \quad & \partial_\mu\left[\tfrac{i}{\sqrt{2}}\partial_a W(\psi^a\sigma^\mu\bar{\theta}) - \tfrac{1}{4}W_a\partial^\mu\phi^a\bar{\theta}^2\right] \end{aligned}$$

Kähler function

More general Lagrangian:

$$\int d^4x\, d^4\theta\ K(\Phi^\dagger, \Phi)$$

SUSY invariance: the SUSY variation of the $\theta^2\bar{\theta}^2$ component of any superfield is a total derivative.

Terminology: the real function K is called the Kähler function.

It gives general kinetic and other (in general non-renormalizable) interactions.

The Real Superfield

Reality is another constraint that can be consistently imposed on a general scalar superfield:

$$V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta})$$

Parametrize the real superfield conveniently

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\chi^\dagger(x) + \tfrac{1}{2}\theta^2[M(x) + iN(x)] \\ &+ \tfrac{1}{2}\bar{\theta}^2[M(x) - iN(x)] + \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta^2[\lambda^\dagger(x) + \tfrac{1}{2}\partial_\mu\chi(x)\sigma^\mu]\bar{\theta} \\ &+ \bar{\theta}^2\theta[\lambda(x) + \tfrac{1}{2}\sigma^\mu\partial_\mu\chi^\dagger(x)] + \tfrac{1}{2}\theta^2\bar{\theta}^2[D(x) - \tfrac{1}{2}\Box C(x)] \end{aligned}$$

Example: real superfield constructed from a chiral superfield Λ

$$\begin{aligned} \Lambda + \Lambda^\dagger &= (A + A^*) + \sqrt{2}\theta\psi + \sqrt{2}\bar{\theta}\psi^\dagger + \theta^2 F + \bar{\theta}^2 F^* - i\theta\sigma^\mu\bar{\theta}\partial_\mu(A - A^*) \\ &+ \tfrac{1}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \tfrac{1}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\psi^\dagger - \tfrac{1}{4}\theta^2\bar{\theta}^2\Box(A + A^*) \end{aligned}$$

Extended gauge invariance

Extended gauge invariance acts on the entire real superfield:

$$V \to V + \Lambda + \Lambda^\dagger$$

It acts as ordinary gauge invariance on the vector field component A_μ:

$$A_\mu \quad \to A_\mu - i\partial_\mu(A - A^*)$$

A SUSY theory invariant under ordinary gauge symmetry is automatically invariant under the full extended gauge symmetry, where the gauge parameter is a chiral superfield, Λ.

The WZ gauge

The WZ-gauge: choose gauge functions of extended gauge invariance so three scalar fields $C(x), M(x), N(x)$ and the Weyl spinors $\chi(x), \chi^\dagger(x)$ are transformed to zero.

The vector supermultiplet in the Wess-Zumino gauge:

$$V^a_{\rm WZ} = \theta\sigma^\mu\bar\theta A^a_\mu + \theta^2\bar\theta\lambda^{\dagger a} + \bar\theta^2\theta\lambda^a + \tfrac{1}{2}\theta^2\bar\theta^2 D^a \ ,$$

Remark: adjoint indices appropriate for non-abelian generalization. The transformation to WZ-gauge is (much) more complicated in that case, but the final result is the same.

Non abelian gauge invariance

Extended gauge transformation:

$$\exp(T^a V^a) \to \exp(T^a \Lambda^{a\dagger}) \exp(T^a V^a) \exp(T^a \Lambda^a)$$

Linear approximation:

$$V^a \to V^a + \Lambda^a + \Lambda^{a\dagger} + \mathcal{O}(V^a \Lambda^a)$$

Higher orders simplify in Wess–Zumino gauge:

$$\begin{aligned} V^a V^b &= \tfrac{1}{2}\theta^2 \bar{\theta}^2 A^{a\mu} A^b_\mu \\ V^a V^b V^c &= 0 \end{aligned}$$

"Field strength" Chiral Superfield

Goal: make extended gauge symmetry manifest, and see that $C(x), M(x), C(x)$ and $\chi(x), \chi^\dagger(x)$ are unphysical (they decouple).

Starting from a vector superfield, define the spinorial superfield

$$T^a W^a_\alpha \quad = \quad -\tfrac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha} e^{-T^a V^a} D_\alpha e^{T^a V^a}$$

Expand in components:

$$W^a_\alpha \quad = \quad \lambda^a_\alpha(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) + \theta^2 i\sigma^\mu D_\mu \lambda^{\dagger a}(y),$$

where

$$\sigma^{\mu\nu} \quad = \quad \tfrac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$$

Terminology: W_α is the "field strength" chiral superfield.

Interpretation: W_α isolates the physical degrees of freedom from the vector superfield (off-shell, the D-term is included).

Why is W_α "chiral"?

Recall: chiral scalar superfields are general superfields satisfying

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

Definition of field strength superfield:

$$T^a W^a_\alpha \quad = \quad -\tfrac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} e^{-T^a V^a} D_\alpha e^{T^a V^a}$$

There are just two distinct super-derivatives $\bar{D}_{\dot{\alpha}}$ ($\dot{\alpha} = \dot{1}$ and $\dot{\alpha} = \dot{2}$) and they "all" anti-commute; so

$$\bar{D}_{\dot{\alpha}} W^a_\alpha = 0$$

Conclusion: W^a_α is chiral.

Corollary: the chiral superfield W^a_α is a function of the superspace coordinate $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$ and θ, but not $\bar{\theta}$.

(Proof: these are the quantities annihilated by $\bar{D}_{\dot{\alpha}}$.)

Component Expansion: SUSY QED

Goal: work out the field strength superfield in components.

Definition:

$$W_\alpha \quad = \quad -\tfrac{1}{4}\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}(D_\alpha V) = D_\alpha V|_{\bar\theta^2 \text{ only}}$$

where the vector field (in WZ-gauge):

$$V_{\rm WZ} = \theta\sigma^\mu\bar\theta A_\mu + \theta^2\bar\theta\lambda^\dagger + \bar\theta^2\theta\lambda + \tfrac{1}{2}\theta^2\bar\theta^2 D \ ,$$

and the superspace derivative (with the y^μ coordinate kept fixed):

$$D_\alpha \quad = \quad \tfrac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar\theta)_\alpha\tfrac{\partial}{\partial y^\mu}$$

Manipulations (details below):

$$\begin{aligned} W_\alpha &= \lambda_\alpha + \theta_\alpha D - 2i(\sigma^\mu\bar\theta)_\alpha[(\theta\sigma^\nu\bar\theta\partial_\mu A_\nu + \theta^2\bar\theta\partial_\mu\lambda^\dagger]_{\bar\theta^2 \text{ only}} \\ &= \lambda_\alpha + \theta_\alpha D - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + i\theta^2(\sigma^\mu\partial_\mu\lambda^\dagger)_\alpha \end{aligned}$$

Reordering:

$$(\sigma^\mu\bar{\theta})_\alpha\theta\sigma^\nu\bar{\theta} = -\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}(\bar{\theta}_{\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\beta}\theta_\beta) = \tfrac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha\bar{\theta}^2$$

where

$$\sigma^{\mu\nu} = \tfrac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$

Some identities:

$$\begin{aligned}
\psi\sigma^\mu\chi^\dagger &= -\chi^\dagger\bar{\sigma}^\mu\psi \\
\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\tfrac{1}{2}\bar{\theta}^2\delta^{\dot{\alpha}}_{\dot{\beta}} \\
(\psi\theta)(\chi\theta) &= -\tfrac{1}{2}(\psi\chi)\theta^2
\end{aligned}$$

Each component field depends on y^μ, the entire W_α depends on y^μ, θ:

$$W_\alpha(y,\theta) = \lambda_\alpha(y) + \theta_\alpha D(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + i\theta^2(\sigma^\mu\partial_\mu\lambda^\dagger(y))_\alpha$$

SUSY Yang–Mills action

Lagranian in superspace notation:

$$\int d^4x\, \mathcal{L}_{SYM} \quad = \quad \tfrac{1}{4} \int d^4x\, d^2\theta\; W^{a\alpha} W^a_\alpha + h.c.$$

where

$$\mathcal{L}_{SYM} = -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\lambda^{\dagger a} \bar{\sigma}^\mu D_\mu \lambda^a + \tfrac{1}{2} D^a D^a$$

An alternate form, as an integral over all of superspace:

$$\int d^4x\, \mathcal{L}_{SYM} \quad = \quad \tfrac{1}{2} \int d^4x\, d^4\theta\; \mathrm{Tr} T^a W^{a\alpha} e^{-T^a V^a} D_\alpha e^{T^a V^a} + h.c.$$

The equivalence:

$$\begin{aligned} T^a W^{a\alpha} &= -\tfrac{1}{4} \bar{D}^2 (e^{-T^a V^a} D_\alpha e^{T^a V^a}) \\ &= [e^{-T^a V^a} D_\alpha e^{T^a V^a}]_{\theta^2 \text{ only}} \\ &= \int d^2\bar{\theta} [e^{-T^a V^a} D_\alpha e^{T^a V^a}] \end{aligned}$$

Matter Couplings

The chiral superfield Φ transforms under the extended gauge invariance as

$$\Phi \to e^{-gT^a \Lambda^a} \Phi$$

Standard gauge invariant kinetic terms:

$$\int d^4\theta \; \Phi^\dagger e^{gT^a V^a} \Phi$$

More generally (to include non-renormalizable interactions):

$$\int d^4\theta \; K(\Phi^\dagger, e^{gT^a V^a} \Phi)$$