Lecture 3
Super Yang-Mills Theory

Outline

- Review: geometry and group theory for gauge theory.
- SUSY Yang-Mills theory: matter content, SUSY transformations, and Lagrangian.
- SUSY QCD: adding matter to Yang-Mills theory.
- The effective scalar potential (the D-term potential).
- SUSY YM couplings in terms of component fields and Feynman diagrams.

Reading: Terning 2.5, 2.6

Geometry of Yang–Mills Theory

Gauge theory is based on a Lie group $\mathbf{G}$ with elements $U(x)$ that depend on spacetime location.

The gauge group transforms matter fields according to some representation $\mathbf{r}$

$$\psi(x) \to U_{\mathbf{r}}(x)\psi(x)$$

The group elements can be written as

$$U_{\mathbf{r}}(x) = e^{ig\Lambda_{\mathbf{r}}(x)} = e^{ig\Lambda^a(x)T^a_{\mathbf{r}}}$$

where the generators T^a in the representation $\mathbf{r}$ satisfy

$$[T^a_{\mathbf{r}}, T^b_{\mathbf{r}}] = if^{abc}T^c_{\mathbf{r}}$$

with f^{abc} the completely antisymmetric structure constants of the Lie algebra. The range of the indices a, b, c is $\dim(\mathbf{g})$, the dimension of the algebra.

Example 1: the adjoint representation

$$(T^b_{\mathbf{Ad}})_{ac} = if^{abc}$$

In this case the dimension of the representation is the same as that of the algebra. For $G = SU(N)$, $\dim(\mathbf{Ad}) = \dim(\mathbf{g}) = N^2 - 1$.

Example 2: the group $G = SU(N)$, with matter in the fundamental representation. In this case the matrices $U(x)$ are the unitary $N \times N$ matrices and the $T^a_{\mathbf{N}}$ are a complete basis of $N \times N$ traceless matrices. Thus $\dim(\mathbf{g}) = N^2 - 1$ but $\dim(\text{fund}) = N$.

The Gauge Connection

Ordinary derivatives involve points that are separated (infinitesimally), so they transform non-locally under gauge transformations. Gauge covariant derivatives are constructed such that they transform covariantly:

$$D_\mu \psi(x) \to U_{\mathbf{r}}(x) D_\mu \psi(x)$$

where

$$D_\mu = \partial_\mu + igA_\mu = \partial_\mu + igA_\mu^a T_{\mathbf{r}}^a$$

This property requires that the gauge connection transforms as

$$\delta_{\text{gauge}} A_\mu = -[D_\mu, \Lambda_{\mathbf{r}}]$$

in the notation where a matter field transforms as

$$\delta_{\text{gauge}} \psi = ig\Lambda_{\mathbf{r}} \psi$$

The field strength

$$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

is constructed so that it transforms in the adjoint,

$$F_{\mu\nu} \rightarrow U(x) F_{\mu\nu} U^\dagger(x)$$

Thus we can form the gauge invariant Yang-Mills action

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{tr}\ F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

In the normalization of the fundamental, where $\mathrm{tr} t^a t^b = \frac{1}{2}\delta^{ab}$.

SUSY Yang–Mills

Reminder: the vector supermultiplet (from the study of the SUSY algebra) has one massless vector particle with two polarizations, and one massless Weyl fermion (the gaugino), also with two physical degrees of freedom.

In the non-abelian case, the gauge particle is in the adjoint representation. The gaugino matter must also be in the adjoint representation, to preserve SUSY .

In component form: a gauge transformation transforms the gauge field (A_μ^a) and the gaugino field (λ^a) as:

$$\begin{aligned} \delta_{\text{gauge}} A_\mu^a &= -\partial_\mu \Lambda^a + g f^{abc} A_\mu^b \Lambda^c \\ \delta_{\text{gauge}} \lambda^a &= g f^{abc} \lambda^b \Lambda^c \end{aligned}$$

where Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling.

Counting degrees of freedom

Gauge field: the vector field has four real components, one is removed by gauge invariance, while the e.o.m. projects out another.

Gaugino: the Weyl spinor has two (complex) components, but half is projected out by the e.o.m.

Summary:

	off-shell	on-shell
A^a_μ	3 d.o.f.	2 d.o.f.
$\lambda^a_\alpha, \lambda^{\dagger a}_{\dot\alpha}$	4 d.o.f.	2 d.o.f.

For SUSY to be manifest off-shell add a real auxiliary boson field D^a.

	off-shell	on-shell
D^a	1 d.o.f.	0 d.o.f.

Note: each of the degrees of freedom counted are in the adjoint of the gauge group.

SUSY Yang-Mills Lagrangian

The proposed SUSY Yang–Mills Lagrangian:

$$\mathcal{L}_{\text{SYM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i\lambda^{\dagger a} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a$$

Comments:

- The gauge field strength is:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

- The gauge covariant derivative of the gaugino is:

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f^{abc} A^b_\mu \lambda^c$$

- The auxiliary field has dimension $[D^a] = 2$

SUSY variations for chiral theory

Reminder: the SUSY transformations for a single chiral supermultiplet,

$$\begin{aligned}\delta\phi &= \epsilon\psi \\ \delta\psi_\alpha &= -i(\sigma^\mu\epsilon^\dagger)_\alpha\,\partial_\mu\phi + \epsilon_\alpha\mathcal{F} \\ \delta\mathcal{F} &= -i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\end{aligned}$$

Comments:

- The boson SUSY variation is essentially the fermion.
- Spinorial SUSY parameter ϵ^α has mass dimension $[\epsilon] = -\frac{1}{2}$.
- The fermion SUSY variation is essentially the boson, with derivative needed for dimensions to work out.
- The auxiliary field $\mathcal{F}$ has mass dimension two because it has no dynamics, so the fermion variation may also have a term that is essentially $\mathcal{F}$, with no derivative acting on it.

- The auxiliary field SUSY variation is essentially the fermion, with derivative needed for dimensions to work out.
- The Pauli matrices $\sigma^\mu_{\alpha\dot{\alpha}}$, $\overline{\sigma}^\mu_{\dot{\alpha}\alpha}$ are contracted with derivatives, to ensure Lorentz invariance.
- The hermitean conjugate SUSY variations are:

$$
\begin{aligned}
\delta\phi^* &= \epsilon^\dagger\psi^\dagger \\
\delta\psi^\dagger_{\dot{\alpha}} &= i(\epsilon\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\phi^* + \epsilon^\dagger_{\dot{\alpha}}\mathcal{F}^* \\
\delta\mathcal{F}^* &= i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon
\end{aligned}
$$

SUSY variations for Yang-Mills

The principles gleaned from the chiral theory essentially determine the SUSY variations for Yang-Mills:

- The SUSY variation transform A_μ^a and λ_α^a into each other.
- The variations should be linear in ϵ and $\epsilon^\dagger$, combined such that A_μ^a is kept real.
- They maintain the correct dimensions of fields with mass dimension of $[\epsilon] = -\frac{1}{2}$.
- The Pauli matrices $\sigma^\mu_{\alpha\dot{\alpha}}$, $\overline{\sigma}^\mu_{\dot{\alpha}\alpha}$ are contracted with Lorentz indices on gradients ∂_μ, to ensure Lorentz invariance.
- The SUSY variation of the gauge field is

$$\delta A_\mu^a = -\frac{1}{\sqrt{2}} \left[\epsilon^\dagger \overline{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \overline{\sigma}_\mu \epsilon \right]$$

- The SUSY variation of λ^a_α involve the derivative of the A^a_μ, but $\partial_\mu A^a_\nu$ is not gauge covariant, unlike λ^a and $F^a_{\mu\nu}$.
- A term proportional to $\epsilon_\alpha D^a$ and no derivatives is allowed by dimensional analysis.
- The SUSY variation of the gaugino and its conjugate are

$$
\begin{aligned}
\delta\lambda^a_\alpha &= -\frac{i}{2\sqrt{2}}(\sigma^\mu\overline{\sigma}^\nu\epsilon)_\alpha\, F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon_\alpha\, D^a \\
\delta\lambda^{\dagger a}_{\dot\alpha} &= \frac{i}{2\sqrt{2}}(\epsilon^\dagger\overline{\sigma}^\nu\sigma^\mu)_{\dot\alpha}\, F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon^\dagger_{\dot\alpha}\, D^a
\end{aligned}
$$

- The SUSY variation of D^a should vanish when the equations of motion are satisfied, so

$$
\delta D^a = -\frac{i}{\sqrt{2}}\left[\epsilon^\dagger\overline{\sigma}^\mu D_\mu\lambda^a - D_\mu\lambda^{\dagger a}\overline{\sigma}^\mu\epsilon\right]
$$

- In each case the numerical factors are determined such that these variations in fact transform the SUSY Yang-Mills Lagrangian into a total derivative.

SUSY Gauge Theories

Goal: couple pure SUSY Yang-Mills to matter fields.

We add chiral supermultiplets $X_j = \phi_j, \psi_j, \mathcal{F}_j$. These are supersymmetric by themselves but their coupling to the gauge fields is nontrivial.

Generally the chiral supermultplets are charged under the gauge field, with the same charge for each member of the chiral supermultiplet

$$\delta_{\mathrm{gauge}} X_j = ig\Lambda^a T^a X_j$$

To preserve gauge invariance, derivatives must then be replaced by gauge covariant derivatives:

$$\begin{aligned} D_\mu \phi_j &= \partial_\mu \phi_j + igA^a_\mu T^a \phi_j \\ D_\mu \phi^{*j} &= \partial_\mu \phi^{*j} - igA^a_\mu \phi^{*j} T^a \\ D_\mu \psi_j &= \partial_\mu \psi_j + igA^a_\mu T^a \psi_j \end{aligned}$$

The introduction of couplings through the gauge covariant derivaties does not by itself preserve SUSY.

New renormalizable interactions:

$$(\phi^* T^a \psi)\lambda^a \ , \quad \lambda^{\dagger a}(\psi^\dagger T^a \phi) \ , \quad (\phi^* T^a \phi) D^a$$

All are required by SUSY (with particular couplings):

- The first two are required to cancel terms in the SUSY transformations of the gauge interactions of ϕ and ψ.
- The third is needed to cancel pieces of the SUSY transformations of the first two terms.

Lagrangian for SUSY gauge theory

$$\mathcal{L} \;=\; \mathcal{L}_{\rm SYM} + \mathcal{L}_{\rm WZ} - \sqrt{2}g\left[(\phi^* T^a \psi)\lambda^a + \lambda^{\dagger a}(\psi^\dagger T^a \phi)\right] + g(\phi^* T^a \phi)D^a.$$

- $\mathcal{L}_{\rm WZ}$ is the general WZ-model, with ordinary derivatives replaced by gauge-covariant derivatives.
- The superpotential in the WZ-model must be gauge invariant:

$$\delta_{\rm gauge} W = ig\Lambda^a \frac{\partial W}{\partial \phi_i} T^a \phi_i = 0.$$

- The SUSY variation of ψ_j have derivatives promoted to gauge co-variant derivatives

$$\delta\psi_{j\alpha} = -i(\sigma^\mu \epsilon^\dagger)_\alpha\, D_\mu \phi_j + \epsilon_\alpha \mathcal{F}_j$$

- The SUSY variation of the auxiliary $\mathcal{F}_j$ has an additional term required by the gaugino interactions:

$$\delta\mathcal{F}_j = -i\epsilon^\dagger \overline{\sigma}^\mu D_\mu \psi_j \;+\; \sqrt{2}g(T^a\phi)_j\, \epsilon^\dagger \lambda^{\dagger a}$$

The Scalar Potential

The auxiliary field D^a enters quadratically and without derivatives so, as for the auxiliary $\mathcal{F}_j$, it can be integrated out exactly by imposing the e.o.m.:

$$D^a = -g\phi^* T^a \phi$$

Upshot: the scalar potential is given by "$\mathcal{F}$-terms" and "D-terms":

$$V(\phi, \phi^*) = \mathcal{F}^{*i}\mathcal{F}_i + \frac{1}{2}D^a D^a = W_i^* W^i + \frac{1}{2}g^2(\phi^* T^a \phi)^2$$

The scalar potential is positive definite (as required by SUSY):

$$V(\phi, \phi^*) \geq 0$$

Condition that vacuum preserves SUSY: $V = 0 \Rightarrow \mathcal{F}_i = 0$ and $D^a = 0$.

Feynman Vertices

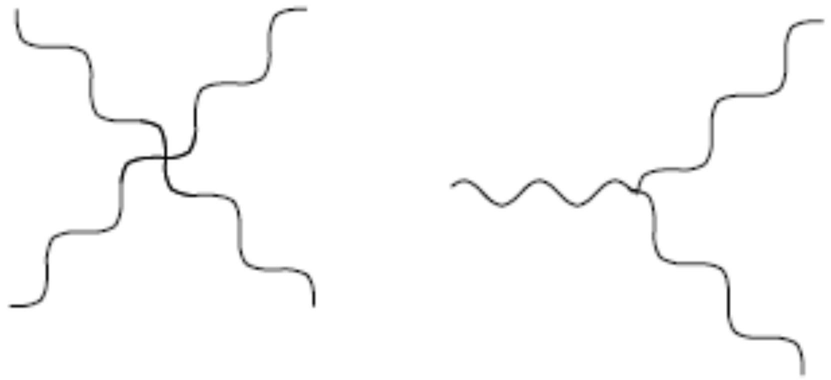

Figure 1:

Cubic and quartic Yang–Mills interactions. Wavy lines denote gauge fields. From $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$.

Figure 2:

Interactions required by gauge invariance. From $\bar{\psi}\not{D}\psi$, $\bar{\lambda}\not{D}\lambda$, $D_\mu\phi D^\mu\phi^*$. Solid lines denote matter fermions, wavy lines denote gauge bosons, wavy/solid lines denote gauginos, dashed lines denote scalars.

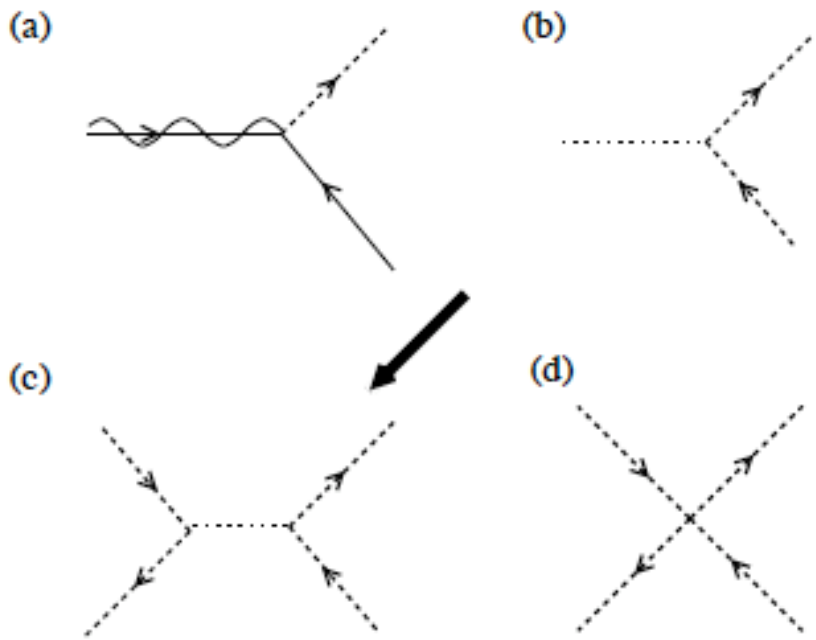

Figure 3:

Additional interactions required by gauge invariance and SUSY: (a) $\phi^*\psi\lambda$, (b) $\phi^*\phi D$ coupling. The (a), (a)*, (b) vertices all have the same gauge index structure (proportional to the gauge generator T^a). Integrating out the auxiliary field in (c) gives (d), the quartic scalar coupling proportional to T^aT^a.

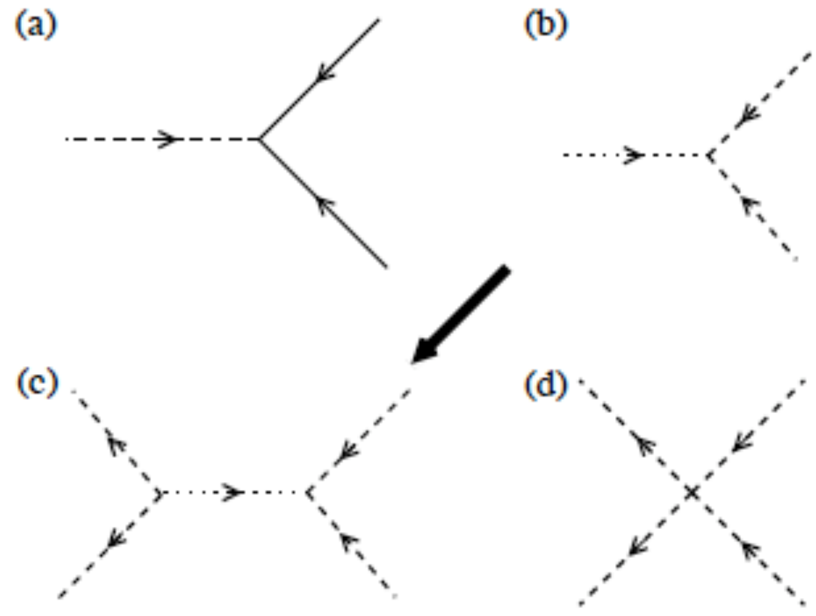

Figure 4:

Dimensionless non-gauge vertices in a supersymmetric theory: (a) $\phi_i \psi_j \psi_k$ Yukawa interaction vertex $-iy^{ijk}$, (b) $\phi_i \phi_j \mathcal{F}_k$ interaction vertex iy^{ijk}, (c) integrating out the auxiliary field yields, (d) the quartic scalar interaction $-iy^{ijn} y^*_{kln}$ (required for cancelling the Λ^2 divergence in the Higgs mass).

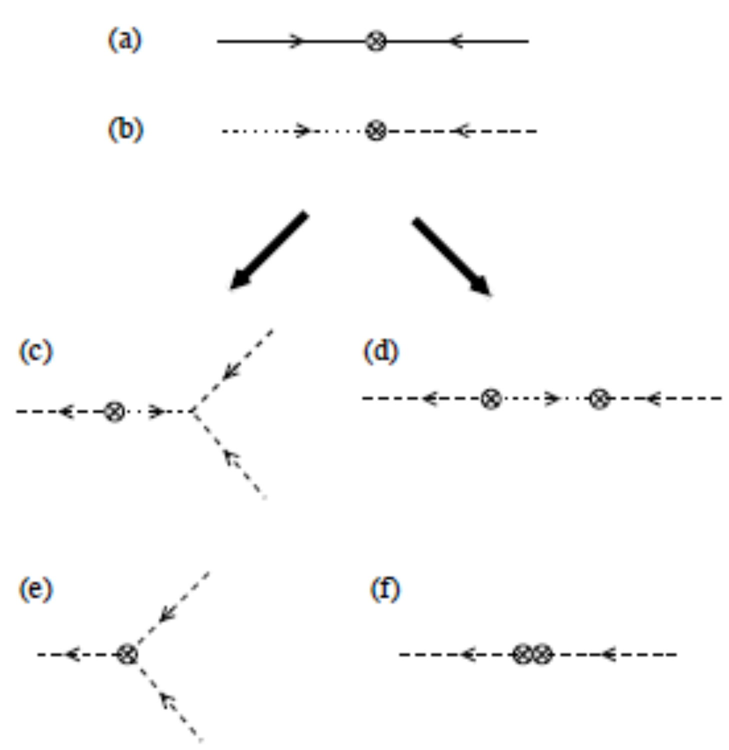

Figure 5:

Dimensionful couplings: (a) $\psi\psi$ mass insertion $-iM^{ij}$, (b) $\phi\mathcal{F}$ mixing term insertion $+iM^{ij}$, (c) integrating out $\mathcal{F}$ in cubic term, (d) integrating out $\mathcal{F}$ in mass term, (e) $\phi^2\phi^*$interaction vertex $-iM^*_{in}y^{jkn}$, (f) $\phi^*\phi$ mass insertion $-iM^*_{ik}M^{kj}$ (ensuring cancellation of $\log\Lambda$ in the Higgs mass).

Supercurrent

The conserved Noether supercurrent, J^μ_α:

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} \quad = \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu X)}\,\delta X - V^\mu$$

For SUSY gauge theory:

$$\begin{aligned} J^\mu_\alpha \quad &= \quad \frac{i}{\sqrt{2}} D^a\,(\sigma^\mu\lambda^{\dagger a})_\alpha + \mathcal{F}_i\,i(\sigma^\mu\psi^{\dagger i})_\alpha \\ &\quad\ +(\sigma^\nu\overline{\sigma}^\mu\psi_i)_\alpha\,D_\nu\phi^{*i} - \frac{1}{2\sqrt{2}}(\sigma^\nu\overline{\sigma}^\rho\sigma^\mu\lambda^{\dagger a})_\alpha\,F^a_{\nu\rho} \end{aligned}$$

It is a long exercise to derive this result.

Status

So far in course:

- Introduced the SUSY algebra (from a simple perspective).
- Introduced the QFTs we will study.
- The simplest general properties of SUSY QFTs.
- Made some reference to Feynman diagrams.

Next few lectures:

- SUSY algebra and SUSY QFTs revisited (from a more general perspective).
- More properties of general SUSY QFTs.
- More experience with Feynman diagrams.