Lecture 25
Superconformal Field Theory
Superconformal Field Theory

Outline:

• The c-theorem.
• The significance of R-symmetry.
• The structure of anomalies in SCFT.
• Computation of R-charges: A-maximization

These Slides are Based on a Guest Lecture by Brian Wecht.
Zamolodchikov’s c-Theorem

The *c*-theorem in 2D states:

1. On the space of QFTs, there exists a functional $c(t, g)$ depending on the RG scale $t = -\ln \mu$ and the couplings $g^I$.
2. Under RG flow to large distance scales $t$, the *c*-function decreases monotonically.
3. At RG fixed points the QFT reduces to a CFT with central charge $c_{\text{CFT}} = c(g^I_*)$.

**Interpretation:** the *c*-function represents the number of degrees of freedom in the QFT. RG-flow decreases it through course-graining. At the fixed points $c$ is equivalent to the degrees of freedom in the CFT.

**Status:** Zamolodchikov proved the *c*-theorem for QFTs in 2D.

Interesting question: does an analogous quantity exist in 4D?
The Trace Anomaly

A clue: in 2D the $c$-function is constructed in terms of the energy-momentum tensor.

For example, 2D CFTs are scale invariant except for the scale anomaly

$$\langle T^\mu_\mu \rangle = -\frac{c}{12} R .$$

Similarly, 4D CFTs are scale invariant except for the trace anomaly:

$$\langle T^\mu_\mu \rangle = a(\text{Euler}) + c(\text{Weyl})^2 + \ldots$$

Idea: $a$ and $c$ depend on the fixed point theory. Might either obey the $c$-theorem?

Evidence: examples of RG flows exist establishing that $c_{4D}$ may increase along the flow $c_{\text{IR}} > c_{\text{UV}}$. Thus $c$ does not satisfy a “$c$-theorem”.

The examples show that any linear combination of $a$ and $c$ similarly increase.
Exception: examples are consistent with \( a \) decreasing along RG-flows.

**Cardy’s conjecture:** \( a \) satisfies a \( c \)-theorem!

Remark: the labelling of \( a \) and \( c \) is historical and has been maintained even though it is \( a \) that satisfies a \( c \)-theorem.

**Status:** there is no proof of the theorem because it is too hard to compute things when they become strongly coupled.

**Strategy:** explore superconformal CFTs where some exact results can be computed.
R-Symmetry

Recall: according to Coleman-Mandula, there is at most one $U(1)$ that does not commute with SUSY $\Rightarrow$ the R-symmetry $U(1)_R$.

Example: a chiral superfield

$$\Phi = \phi + \theta \psi + \theta^2 F,$$

$$\phi \rightarrow e^{ir\alpha}, \psi \rightarrow e^{i(r-1)\alpha}, \phi \rightarrow e^{i(r-2)\alpha},$$

Remarks:

- R-symmetry is not part of the SUSY algebra so it may not be a symmetry even though SUSY is a symmetry.

- It is often useful to implement R-symmetry as a spurious symmetry (that acts on couplings as well), even when it is not a symmetry.

- R-symmetry is a part of the superconformal algebra $SU(2,2|1) \supset SO(4,2) \times U(1)_R$, so for SCFTs it always applies.
Consequences of R-symmetry

The scaling dimension is determined exactly for chiral primary fields:

\[ \Delta(\mathcal{O}) \geq \frac{3}{2} |R(\mathcal{O})| . \]

Consequence: anomalous dimension for chiral primaries

\[ \Delta(\mathcal{O}) = 1 + \frac{1}{2} \gamma(\mathcal{O}) \Rightarrow \gamma(\mathcal{O}) = 3R(\mathcal{O} - 2) . \]

The NSVZ \( \beta \)-function

\[ \beta_{NSVZ} \sim 3T(G) - \sum T(r_i)(1 - \gamma_i) = 3[T(Ad) + \sum_i T(r_i)(R_i - 1)] \]

Note: the RHS is precisely the anomaly in the R-current, since \( R(\text{gaugino}) = 1, R(\text{sfermion}) = R_i - 1. \)

So: \( \beta_{NSVZ} = 0 \) (scale invariance) \( \Leftrightarrow \partial_\mu R^\mu = 0 \) (conserved R-current).

**Interpretation:** the scale current and the R-current are both part of the SCA so their anomalies are proportional.
The precise connection employs the “super stress tensor”:

\[ T_{\alpha\dot{\alpha}}(x, \theta, \bar{\theta}) = R_{\alpha\dot{\alpha}} + \theta^\beta S_{\alpha\dot{\alpha}\beta} + \theta^\beta \theta^{\dot{\beta}} T_{\alpha\dot{\alpha}\beta\dot{\beta}} + \cdots . \]

The superfield anomaly equation is

\[ \nabla^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \nabla_\alpha L , \]

where

\[ L = \frac{c}{24\pi^2} \mathcal{W}^2 - \frac{a}{24\pi^2} \Xi - \frac{1}{96\pi^2} \sum_{I,J} \tau_{IJ} \text{Tr} W_I W_J , \]

Notation: \( \mathcal{W}^2 = \text{super Weyl}^2 \), \( \Xi = \text{Super Euler} \), \( \mathcal{W}_I^\alpha = \text{spinor super field strength} \).

The super-curvature terms on the RHS include both conventional Riemann curvature and, by SUSY, the \( U(1)_R \) gauge fields.
Structure of Anomalies

One component of the superfield anomaly equation expresses the trace anomaly in the form

\[ T_\mu^\mu = c(\text{Weyl})^2 + a(\text{Euler}) + \ldots , \]

while another (related to the first by SUSY) expresses the R-charge anomaly in the form

\[ \partial_\mu R^\mu = (\ldots a + \ldots c)\text{Riem} \cdot \text{Riem} + (\ldots a + \ldots c)F_R \cdot \tilde{F}_R + \ldots . \]

Interpretation in terms of triangle diagrams:

- Anomaly coefficient in front of \( \text{Riem} \cdot \text{Riem} \): \( U(1)_R \) at one angle, and gravitons at the two others \( \Rightarrow \) this coefficient is proportional to \( \text{Tr} R \).

- Anomaly coefficient in front of \( F_R \cdot \tilde{F}_R \): \( U(1)_R \) at one angle, and \( U(1)_R \) gauge fields at the two others \( \Rightarrow \) this coefficient is proportional to \( \text{Tr} R^3 \).
\( a \) and \( c \) from \( R \)

Detailed computations of the coefficients give:

\[
\begin{align*}
    a &= \frac{3}{32} \left( 3 \text{Tr} R^3 - \text{Tr} R \right), \\
    c &= \frac{1}{32} \left( 9 \text{Tr} R^3 - 5 \text{Tr} R \right).
\end{align*}
\]

Status: we can compute the \( a \) and \( c \) coefficients from the \( R \) charges of the chiral fields.

To do: determine the R-charges.

Reminder: the R-charge is completely specified in principle by the SCA, so we should encounter no ambiguity.
Computation of $R$-charges

The procedure applied in the course so far determined the $R$-charges by the consistency conditions:

1. The $R$-current anomaly must cancel.
2. The $R$ charge of the superpotential $R(W) = 2$.

Example (standard SUSY QCD): $SU(N_c)$ gauge theory with $N_f$ flavors $Q, \tilde{Q}$ in the fundamental of $SU(N_c)$ and no superpotential.

Anomaly cancellation:

$$2 T(\text{Ad}) + 2 N_f \cdot 2 T(\square) \cdot (R(Q) - 1) = 0 \implies R(Q) = 1 - \frac{N_c}{N_f}.$$
Ambiguities in the $R$-charges

In more complicated examples these conditions are not sufficient to determine the $R$-charge.

**Example:** standard SUSY QCD (as above), still no superpotential, but add an additional field $X$ in the adjoint of the $SU(N_c)$.

The condition for anomaly cancellation acquires an additional term, preventing a unique solution:

$$2T(\text{Ad}) + 2N_f \cdot 2T(\Box) \cdot (R(Q) - 1) + 2T(\text{Ad}) \cdot (R(X) - 1) = 0.$$ 

**Alternative Formulation of the Ambiguity:** suppose that there is at least one anomaly free $U(1)$ current (with superpotential charge $F_I(W) = 0$) in addition to the proposed $R$-current with charges $R_0$.

Then there is a family of consistent assignments for the superconformal $R$-charge:

$$R = R_0 + \sum_I s_I F_I.$$
More Consistency Conditions

The R-current in the SCA is unique $\Rightarrow$ there must be additional consistency conditions that determine $s_I$.

There are two new ones:

3. $9 \text{Tr} R^2 F_I = \text{Tr} F_I$.
4. $\text{Tr} R F_I F_J < 0$. 
Sketch of Proof

The superconformal symmetry relates different triangle anomalies: \( \langle J_R J_R J_i \rangle \) is related to \( \langle TT J_i \rangle \) by

\[
9 \text{Tr} R^2 Q_i = \text{Tr} Q_i
\]

two-point function

\[
\langle J_i(x) J_k(0) \rangle \propto \tau_{ik} \frac{1}{x^4}
\]

Unitarity \( \Rightarrow \tau_{ik} \) to have positive definite eigenvalues

Superconformal symmetry \( \Rightarrow \)

\[
\text{Tr} RQ_i Q_k = -\frac{\tau_{ik}}{3}
\]

\( \Rightarrow \text{Tr} RQ_i Q_k \) is negative definite
A-Maximization

The additional two conditions are equivalent to maximizing the $a$-charge:

$$a_{\text{trial}} = \frac{3}{32} \left( 3 \text{Tr} R_{\text{trial}}^3 - \text{Tr} R_{\text{trial}} \right)$$

where the trial $R$-charges are

$$R_{\text{trial}} = R_0 + \sum_I s_I Q_I .$$

This procedure is known as $a$-maximization.

Sanity check: a single chiral superfield $\Phi$ with unknown $R(\Phi) = r$.

$$a_{\text{trial}} = \frac{3}{32} \left( 3(r - 1)^3 - (r - 1) \right) \Rightarrow \frac{da_{\text{trial}}}{dr} = 9(r - 1)^2 - 1 \Rightarrow r = \frac{2}{3}, \frac{4}{3} .$$

The alternate root $r = \frac{4}{3}$ is a local minimum.
Detailed Example
The C-Theorem Revisited

Reminder: the conjectured $c$-theorem posits that $a_{\text{IR}} < a_{\text{UV}}$ for an RG flow.

$a$-maximisation hints at a proof of the $c$-theorem:

- Begin with some CFT in the UV and deform by a relevant operator $\mathcal{O}$.

- The operator generally break some $U(1)$ symmetries, so the space of symmetries is smaller after the flow that before:

$$\mathcal{S}_{\text{IR}} \subset \mathcal{S}_{\text{UV}}.$$

- $a$-maximization over the smaller set $\mathcal{S}_{\text{IR}}$ generally give a smaller maximum that $a$-maximization over the larger set $\mathcal{S}_{\text{UV}} \Rightarrow a_{\text{IR}} < a_{\text{UV}}.$
Loopholes

The argument for the C-theorem remains just a hint since there are substantial loopholes:

1. **Accidental symmetries**: it is not necessarily true that $S_{\text{IR}} \subset S_{\text{UV}}$ since new “accidental” symmetries can appear in the IR. This is an extremely difficult issue to circumvent since not much is known about accidental symmetries.

2. **Local maximum** is not sufficient: the maximum required by the $a$-maximization is generally a local maximum, not a global maximum. So it is possible, upon maximization over the smaller set, “another” maximum dominates.

This problem has been largely addressed by Kutasov et.al., by constructing an extremization procedure that is local along the flow, thus connecting the maxima of the UV and the IR.

**Status**: there is still no proof of the c-theorem in 4D.