Lecture 11
Anomalies and Instantons

Outline

- The chiral anomaly: detailed discussion.
- Zero-modes and the index theorem.
- Triangle diagrams: the qualitative story.
- The gauge anomaly.
- Anomaly matching.

Reading: Terning 7.2, 7.3, 7.4, 7.5.

The Chiral Anomaly

Setting: massless chiral fermions coupled to a classical gauge field.

Comment: for *massless* particles chiralities make sense – we can focus on two component Weyl spinors (”the upper two components” of a Dirac four-spinor).

Lagrangian:

$$\mathcal{L}_{\text{fermion}} = i\bar{\psi}\bar{\sigma}^\mu D_\mu \psi$$

Notation:

- $\bar{\psi} = \psi^\dagger$ for two component spinors.
- $\bar{\sigma}^\mu = (1, -\vec{\sigma})$.
- $D_\mu = \partial_\mu + iB_\mu$ with $B_\mu = B_\mu^a T^a$ a gauge field with values in a Lie algebra.

Consider the fermion path integral as function of a background gauge field:

$$Z[B_\mu] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{iS_{\text{fermion}}} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{-S_E} = \det(\not{D})\ .$$

Euclidean continuation:

- $\not{D} = i\bar{\sigma}_E^\mu(\partial_\mu + iB_\mu)$ with $\bar{\sigma}_E^\mu = (iI_2, -\vec{\sigma})$.
- $id^4x = d^4x_E$.
- $S_E = -\int d^4x_E\bar{\psi}\ \not{D}\psi$.

More precisely: expand ψ, $\bar{\psi}$ on complete bases

$$\psi(x) = \sum_n a_n f_n(x) \quad , \; \bar{\psi}(x) = \sum_n b_n g_n^\dagger(x) \; .$$

Orthogonality and completeness:

$$\sum_n f_n(x) f_n^\dagger(y) = \delta(x-y) I_2 \quad , \; \sum_n g_n(x) g_n^\dagger(y) = \delta(x-y) I_2 \; .$$

$$\mathrm{Tr} \int d^4x f_n^\dagger(x) f_m(x) = \delta_{nm} \quad , \; \mathrm{Tr} \int d^4x g_n^\dagger(x) g_m(x) = \delta_{nm} \; .$$

Choose basis so for some real λ_n's:

$$\not{D} f_n = \lambda_n g_n \; .$$

Then the path-integral is a diagonal Gaussian:

$$Z[B_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_E} = \int \prod_{n,m} da_n db_m e^{\sum_n \lambda_n b_n a_n} = \prod_n \lambda_n \; .$$

More on eigenvalues λ_n and bases $\{f_n\}$, $\{g_n\}$.

Issue: the Dirac operator $\not{D}$ maps positive chirality spinors to negative chirality spinors. Therefore it can have no eigenfunctions: input and output is not even on the same space!

Resolution: $\not{D}^\dagger \not{D}$ is a hermitean operator on the positive chirality space so there is a complete basis of eigenfunctions:

$$\not{D}^\dagger \not{D} f_n = \lambda_n^2 f_n \ ,$$

with λ_n^2 real and non-negative.

The operator with opposite ordering has eigenfunctions:

$$\not{D} \not{D}^\dagger g_n = \lambda_n^2 g_n \ .$$

The two orderings are related as $g_n = \frac{1}{\lambda_n} \not{D} f_n$ (or $f_n = \frac{1}{\lambda_n} \not{D}^\dagger g_n$) so the operators have the same spectrum *except* for zero-modes, ie. the eigenvalues $\lambda_n = 0$:

$$\not{D} f_n = 0 \quad or \quad \not{D}^\dagger g_n = 0 \ .$$

The Chiral Current

The Noether current for classical symmetry under $\psi(x) \rightarrow e^{i\alpha}\psi(x)$, $\bar{\psi}(x) \rightarrow e^{-i\alpha}\bar{\psi}(x)$.

Remark: this is a *chiral* transformation because $\psi(x)$ is a Weyl fermion (upper two components only).

At the quantum level: change variables in the path integral as

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \quad , \; \bar{\psi}(x) \rightarrow e^{-i\alpha(x)}\bar{\psi}(x) \; .$$

The classical action transforms as

$$S_E \rightarrow S_E - \int d^4x_E \bar{\psi} e^{-i\alpha(x)} i\bar{\sigma}^\mu(\partial_\mu e^{i\alpha})\psi = S_E - \int d^4x_E \alpha(x)\partial_\mu\left(\bar{\psi}\bar{\sigma}^\mu\psi\right) \; .$$

The classical action is invariant under *arbitrary* variations so the current is conserved

$$\partial_\mu j_A^\mu = 0 \quad , \; j_A^\mu(x) = \bar{\psi}\bar{\sigma}^\mu\psi \; .$$

The chiral anomaly: quantum correction to classical current conservation.

The chiral transformation acts on the basis coefficients (Tr=trace over spin and group indices):

$$\psi(x) = \sum_n a_n f_n(x) \to \sum_n a_n e^{i\alpha(x)} f_n(x) = \sum_n a'_n f_n(x) \ ,$$

$$a_n \to a'_n = C_{nm} a_m = \mathrm{Tr} \int d^4x_E f_n^\dagger(x) e^{i\alpha(x)} f_m(x) a_m \ .$$

The path integral measure transforms accordingly:

$$\prod_{n,m} da_n db_m \to (\det C\bar{C})^{-1} \prod_{n,m} da_n db_m \ .$$

where

$$(\det C\bar{C})^{-1} = e^{-i\int d^4x_E \alpha(x) A(x)} \ , \ A(x) = \mathrm{Tr} \sum_n (f_n^\dagger(x) f_n(x) - g_n^\dagger(x) g_n(x)) \ .$$

Quantum equation for anomalous current

$$\partial_\mu j_A^\mu = A(x) \ .$$

Compute anomaly using completeness (regulate by damping large λ_N^2):

$$\begin{aligned}
A(x) &= \mathrm{Tr}\sum_n (f_n^\dagger(x) f_n(x) - g_n^\dagger(x) g_n(x)) \\
&= \lim_{\Lambda\to\infty} \mathrm{Tr}\sum_n [f_n^\dagger(x) e^{-\lambda_n^2/\Lambda^2} f_n(x) - g_n^\dagger(x) e^{-\lambda_n^2/\Lambda^2} g_n(x)] \\
&= \lim_{\Lambda\to\infty} \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4} [e^{-\not{D}^\dagger \not{D}/\Lambda^2} - e^{-\not{D}\not{D}^\dagger/\Lambda^2}] \\
&= \lim_{\Lambda\to\infty} \int \frac{d^4p}{(2\pi)^4\Lambda^4} e^{-D_\mu D^\mu/\Lambda^2} \mathrm{Tr}[\frac{1}{2}(\sigma_E^{\mu\nu} F_{\mu\nu})^2 - \frac{1}{2}(\bar{\sigma}_E^{\mu\nu} F_{\mu\nu})^2] \\
&= \int \frac{\pi^2 p^2 dp^2}{(2\pi)^4} e^{-p^2} \cdot \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \mathrm{Tr} F_{\mu\nu} F_{\rho\lambda} = \frac{1}{16\pi^2} \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \ .
\end{aligned}$$

Explicit form of Dirac operators and Pauli matrices:

$$\not{D}^\dagger \not{D} = D_\mu D^\mu + \frac{i}{4}(\sigma_E^\mu \bar{\sigma}_E^\nu - \sigma_E^\nu \bar{\sigma}_E^\mu) i [D_\mu, D_\nu] = D_\mu D^\mu - \sigma_E^{\mu\nu} F_{\mu\nu}$$

$$\not{D} \not{D}^\dagger = D_\mu D^\mu + \frac{i}{4}(\bar{\sigma}_E^\mu \sigma_E^\nu - \bar{\sigma}_E^\nu \sigma_E^\mu) i [D_\mu, D_\nu] = D_\mu D^\mu - \bar{\sigma}_E^{\mu\nu} F_{\mu\nu}$$

$$\mathrm{Tr}[\sigma_E^{\mu\nu} \sigma_E^{\rho\lambda} - \bar{\sigma}_E^{\mu\nu} \bar{\sigma}_E^{\rho\lambda}] = \epsilon^{\mu\nu\rho\lambda}$$

Summary: the operator form of the chiral anomaly:

$$\partial_\mu j^\mu_A = A(x) = \frac{1}{16\pi^2} \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \ .$$

The important point: current conservation at the classical level is violated by a quantum effect of a very specific form.

The Atiyah-Singer Index Theorem

Instructive alternative computation of $A(x)$:

$$g_n = \frac{1}{\lambda_n} \not{D} f_n \quad , \quad f_n = \frac{1}{\lambda_n} \not{D}^\dagger g_n \ ,$$

so contributions to $A(x)$ cancel in pairs except for $\lambda_n = 0$:

$$\int d^4x A(x) = \mathrm{Tr} \int d^4x \sum_n (f_n^\dagger(x) f_n(x) - g_n^\dagger(x) g_n(x)) = n_\psi - n_{\bar{\psi}} \ .$$

The number of zero modes of $\not{D}$, $\not{D}^\dagger$: $n_\psi, n_{\bar{\psi}}$.

The Atiyah-Singer index theorem:

$$n_\psi - n_{\bar{\psi}} = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \ .$$

Mathematical significance: a relation between a topological quantity (the difference in the number of zero-modes cannot change continuously) to an analytical quantity (an integral over a smooth function).

Towards the physical significance: there would be no chiral anomaly were it not for the zero modes of the Dirac operator (and those make the path integral vanish).

Instantons

The Euclidean action $S \sim \int d^4x \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$ of a gauge field configuration diverges at infinity, *unless* the gauge field is asymptotically pure gauge $A_\mu \to U(x) \partial_\mu U(x)^\dagger$.

$\mathrm{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$ is a total derivative so its integral vanishes *except* for a surface term at the asymptotic Euclidean 3-sphere S^3. This surface term counts the wrapping number ν of the $U(x)$ around the asymptotic S^3.

Example: $G = SU(2) \sim S^3$ as a group, so the obvious embedding $G \to S^3$ has winding number $\nu = 1$.

Field configurations with winding number have non-trivial field strength in bulk. Their Euclidean action is finite but large for small coupling $S_E \sim 1/g^2$.

These configurations are localized in Euclidean space, including in time. They are called instantons.

The explicit integral over a winding configuration gives

$$\frac{1}{16\pi^2}\int d^4x \mathrm{Tr} F_{\mu\nu}\tilde{F}_{\rho\lambda} = \nu \ .$$

Remark: the winding number ν is just the integral difference of zero modes $n_\psi - n_{\bar{\psi}}$.

The integral above is in the fundamental representation (denoted $\square$) where the gauge matrices $U(x)$ are just the defining matrices.

The index $T(\square) = \frac{1}{2}$ in a general representation $\mathbf{r}$ is

$$\mathrm{Tr}[T^a T^b] = T(\mathbf{r})\delta^{ab} \ .$$

The anomaly for a theory with $n(\mathbf{r})$ chiral fermions in representation $\mathbf{r}$:

$$\int d^4x A(x) = \frac{1}{16\pi^2}\int d^4x \mathrm{Tr} F_{\mu\nu}\tilde{F}^{\mu\nu} = \sum_r n(\mathbf{r}) 2T(\mathbf{r}) \ .$$

The θ-term

The gauge theory action generally includes the CP-violating term

$$S_{\rm CP} = -\tfrac{\theta_{\rm YM}}{16\pi^2} \int d^4x {\rm Tr} F^{\mu\nu} \tilde{F}_{\mu\nu} \ .$$

The term is a total derivative but it does not vanish:

$$S_{\rm CP} = -\theta_{\rm YM} \nu \ .$$

The path integral sums over *all* field configurations, with different instanton sectors weighed by the phase factor $e^{-i\theta_{\rm YM}\nu}$.

Theories with different $\theta_{\rm YM}$ are generally physically distinct. Specifically they have different spectrum.

Exception: shifting the phase $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi\mathbb{Z}$ is inconsequential.

The chiral transformation $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$ shifts the path integral as:

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS} \to \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS-i\alpha\int d^4xA(x)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS-i\frac{\alpha}{16\pi^2}\int d^4x\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}} .$$

Equivalently:

$$\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} - \alpha\sum_r n_r 2T(\mathbf{r}) .$$

Example: in pure SYM the gaugino is the only chiral fermion. A chiral rotation $\lambda^a \to e^{i\alpha}\lambda^a$ is equivalent to

$$\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} - 2N\alpha .$$

because the index of the adjoint is $T(\mathbf{Ad}) = N$.

Thus the chiral anomaly breaks the continuous $U(1)_R$ in the classical theory to the Z_{2N} generated by $\alpha = \frac{k}{N}\pi$ (with $k = 0, \ldots, 2N-1$).

The Euclidean action of instantons is bounded:

$$0 \leq \int d^4x \mathrm{Tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 = 2 \int d^4x \left(F_{\mu\nu}F^{\mu\nu} \pm F_{\mu\nu}\tilde{F}^{\mu\nu} \right) ,$$

so

$$\int d^4x F_{\mu\nu}F^{\mu\nu} \leq \left| \int d^4x F_{\mu\nu}\tilde{F}^{\mu\nu} \right| = 16\pi^2\nu .$$

Instantons in fact saturate this bound.

Instanton contributions to the Euclidean path integral are suppressed by the characteristic nonperturbative factor:

$$\left(e^{-\frac{8\pi^2}{g^2}+i\theta} \right)^{\nu} .$$

Such effects are qualitatively important because they are the leading contributions to some processes.

For example, we would like to break SUSY in a non-perturbative manner, where the breaking is heavily suppressed.

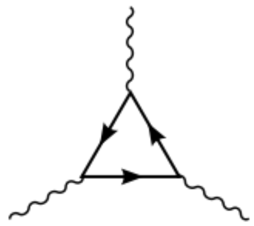
Triangle Diagrams

Alternative derivation of chiral anomaly: Triangle diagrams.

Chiral current (in the interacting theory):

$$\langle j_A^\mu(x)\rangle_{\text{int}} \quad = \quad \langle\bar{\psi}(x)\gamma^\nu\gamma^5\psi(x)\; e^{-ie\int d^4y\bar{\psi}\not{A}\psi}\rangle_0$$

Expanding to second order, contract all fermions: find triangle diagram with current insertion at one angle and gauge field on two others:



The divergence of the chiral current vanishes formally:

$$q_\mu\langle j_A^\mu(q)\rangle_{\text{int}} = 0 \quad \text{(formally)}\ .$$

The reason: explicit integrals for the two diagrams cancel upon linear shifts in the loop momentum variables.

The catch: shift of loop momentum is only justified if the regulated integral preserves the shift symmetry. This is not possible in any regularization scheme.

Explicit computations in several regularization schemes

$$\partial_\mu \langle j_A^\mu \rangle_{\rm int} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \ ,$$

The Gauge Anomaly

Setting: a chiral gauge theory with matter in a chiral representation (the couplings are chiral).

Sample Lagrangian:

$$\mathcal{L} = \psi_L^\dagger i\overline{\sigma}^\mu D_\mu \psi_L + \psi_R^\dagger i\sigma\partial_\mu\psi_R \ , \quad D_\mu = \partial_\mu - igA_\mu^a t^a$$

Noether current of interest: the gauge current (the symmetry is not chiral):

$$j^{\mu a} = \psi_L \overline{\sigma}^\mu t^a \psi_L$$

Gauge anomaly: two triangle diagrams fail to cancel (but details quite different from the chiral anomaly).

The significance: the gauge current must be exactly conserved or else the decoupling of longitudinal gauge particles (and ghosts) fail.

Gauge theory is inconsistent at the quantum level if there is a gauge anomaly.

Evaluation:
1) explicit computation of triangle diagram.
or
2) topological argument (reducing path integral to zero modes).

Result at linear order (non-linear order not just field strength!):

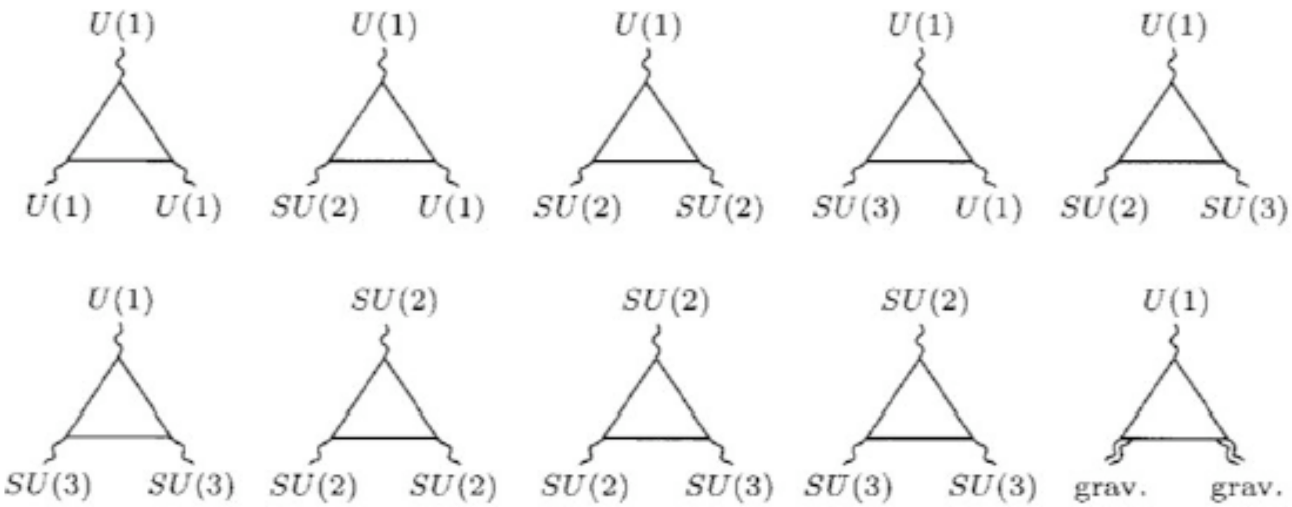
$$\partial_\mu \langle j^{\mu a} \rangle = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F^b_{\mu\nu} F^c_{\alpha\beta} \mathcal{A}^{abc} \ ,$$

where the anomaly coefficient:

$$\mathcal{A}^{abc} = \mathrm{tr}[t^a, \{t^b, t^c\}] \ .$$

Consistency condition on gauge theory: the anomaly coefficient must vanish (when summed over all matter representations).

Anomaly Cancellation in SM



Remarkable fact: SM spectrum is such that all gauge anomalies vanish.

Example: anomaly of $U(1)_{\mathrm{EM}} \times SU(2)_L^2$:

$$\mathrm{tr}[Q, \{\tau^a, \tau^b\}] = \delta^{ab}\mathrm{tr}[Q] = 3 \cdot (\tfrac{2}{3} - \tfrac{1}{3}) + (0 - 1) = 0 \ .$$

Anomaly Matching

Important application of anomalies: 't Hooft's anomaly matching.

Setting: a UV theory with some global symmetry G. Anomaly matching gives information about the IR behavior of the theory.

Promote G to a gauge symmetry, with some weak coupling g.

The anomaly coefficients A_{UV} may fail to vanish. Add some spectator matter fields that only couple to G, designed such that $A_{UV} + A_S = 0$.

The gauge group G is just a redundancy of the theory so it must remain in the IR theory.

The anomaly coefficients (including the contribution from the spectators) of the IR theory therefore must vanish $A_{IR} + A_S = 0$. Thus $A_{UV} = A_{IR}$.

The gauge bosons and the spectator fields decouple as $g \to 0$ but maintain the three point functions of the currents so that $A_{UV} = A_{IR}$ remains.