Lecture 11
Anomalies and Instantons
Outline

• The chiral anomaly: detailed discussion.
• Zero-modes and the index theorem.
• Triangle diagrams: the qualitative story.
• The gauge anomaly.
• Anomaly matching.

Reading: Terning 7.2, 7.3, 7.4, 7.5.
The Chiral Anomaly

Setting: massless chiral fermions coupled to a classical gauge field.

Comment: for *massless* particles chiralities make sense – we can focus on two component Weyl spinors ("the upper two components" of a Dirac four-spinor).

Lagrangian:

\[ \mathcal{L}_{\text{fermion}} = i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi \]

Notation:

- \( \bar{\psi} = \psi^\dagger \) for two component spinors.
- \( \bar{\sigma}^\mu = (1, -\vec{\sigma}) \).
- \( D_\mu = \partial_\mu + iB_\mu \) with \( B_\mu = B^a_\mu T^a \) a gauge field with values in a Lie algebra.
Consider the fermion path integral as function of a background gauge field:

\[ Z[B_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{iS_{\text{fermion}}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E} = \det(\mathcal{D}) \ . \]

Euclidean continuation:

- \( \mathcal{D} = i\bar{\sigma}_E^\mu (\partial_\mu + iB_\mu) \) with \( \bar{\sigma}_E^\mu = (iI_2, -\bar{\sigma}) \).
- \( id^4x = d^4x_E \).
- \( S_E = -\int d^4x_E \bar{\psi} \mathcal{D}\psi \).
More precisely: expand $\psi$, $\bar{\psi}$ on complete bases

$$\psi(x) = \sum_n a_n f_n (x) \quad , \quad \bar{\psi}(x) = \sum_n b_n g_n^\dagger (x) .$$

Orthogonality and completeness:

$$\sum_n f_n (x) f_n^\dagger (y) = \delta (x - y) I_2 \quad , \quad \sum_n g_n (x) g_n^\dagger (y) = \delta (x - y) I_2 .$$

$$\text{Tr} \int d^4x f_n^\dagger (x) f_m (x) = \delta_{nm} \quad , \quad \text{Tr} \int d^4x g_n^\dagger (x) g_m (x) = \delta_{nm} .$$

Choose basis so for some real $\lambda_n$’s:

$$\mathcal{D} f_n = \lambda_n g_n .$$

Then the path-integral is a diagonal Gaussian:

$$Z[B_\mu] = \int \mathcal{D} \psi \mathcal{D} \bar{\psi} e^{-SE} = \int \prod_{n,m} da_n db_m e^{\sum_n \lambda_n b_n a_n} = \prod_n \lambda_n .$$
More on eigenvalues $\lambda_n$ and bases $\{f_n\}$, $\{g_n\}$.

Issue: the Dirac operator $\mathcal{D}$ maps positive chirality spinors to negative chirality spinors. Therefore it can have no eigenfunctions: input and output is not even on the same space!

Resolution: $\mathcal{D}^\dagger \mathcal{D}$ is a hermitean operator on the positive chirality space so there is a complete basis of eigenfunctions:

$$\mathcal{D}^\dagger \mathcal{D} f_n = \lambda_n^2 f_n,$$

with $\lambda_n^2$ real and non-negative.

The operator with opposite ordering has eigenfunctions:

$$\mathcal{D} \mathcal{D}^\dagger g_n = \lambda_n^2 g_n.$$

The two orderings are related as $g_n = \frac{1}{\lambda_n} \mathcal{D} f_n$ (or $f_n = \frac{1}{\lambda_n} \mathcal{D}^\dagger g_n$) so the operators have the same spectrum except for zero-modes, ie. the eigenvalues $\lambda_n = 0$:

$$\mathcal{D} f_n = 0 \quad \text{or} \quad \mathcal{D}^\dagger g_n = 0.$$
The Chiral Current

The Noether current for classical symmetry under \( \psi(x) \to e^{i\alpha} \psi(x) \), \( \bar{\psi}(x) \to e^{-i\alpha} \bar{\psi}(x) \).

Remark: this is a chiral transformation because \( \psi(x) \) is a Weyl fermion (upper two components only).

At the quantum level: change variables in the path integral as

\[
\psi(x) \to e^{i\alpha(x)} \psi(x) \quad , \quad \bar{\psi}(x) \to e^{-i\alpha(x)} \bar{\psi}(x) .
\]

The classical action transforms as

\[
S_E \to S_E - \int d^4x E \bar{\psi} e^{-i\alpha(x)} i\bar{\sigma}^\mu (\partial_\mu e^{i\alpha}) \psi = S_E - \int d^4x E \alpha(x) \partial_\mu (\bar{\psi} \bar{\sigma}^\mu \psi) .
\]

The classical action is invariant under arbitrary variations so the current is conserved

\[
\partial_\mu j_A^\mu = 0 \quad , \quad j_A^\mu(x) = \bar{\psi} \bar{\sigma}^\mu \psi .
\]
The chiral anomaly: quantum correction to classical current conservation.

The chiral transformation acts on the basis coefficients (Tr=trace over spin and group indices):

\[ \psi(x) = \sum_n a_n f_n(x) \rightarrow \sum_n a_n e^{i\alpha(x)} f_n(x) = \sum_n a'_n f_n(x) , \]

\[ a_n \rightarrow a'_n = C_{nm} a_m = \text{Tr} \int d^4x E f_n^\dagger(x) e^{i\alpha(x)} f_m(x) a_m . \]

The path integral measure transforms accordingly:

\[ \prod_{n,m} da_n db_m \rightarrow (\text{det} C C^\dagger)^{-1} \prod_{n,m} da_n db_m . \]

where

\[ (\text{det} C C^\dagger)^{-1} = e^{-i \int d^4x E \alpha(x) A(x)} , \quad A(x) = \text{Tr} \sum_n (f_n^\dagger(x) f_n(x) - g_n^\dagger(x) g_n(x)) . \]

Quantum equation for anomalous current

\[ \partial_{\mu} j_{A}^\mu = A(x) . \]
Compute anomaly using completeness (regulate by damping large $\lambda^2_N$):

$$A(x) = \text{Tr} \sum_n (f_n^\dagger(x)f_n(x) - g_n^\dagger(x)g_n(x))$$

$$= \lim_{\Lambda \to \infty} \text{Tr} \sum_n [f_n^\dagger(x)e^{-\lambda^2_n/\Lambda^2}f_n(x) - g_n^\dagger(x)e^{-\lambda^2_n/\Lambda^2}g_n(x)]$$

$$= \lim_{\Lambda \to \infty} \text{Tr} \int \frac{d^4p}{(2\pi)^4} [e^{-\slashed{p}^\dagger \slashed{p}/\Lambda^2} - e^{-\slashed{p}\slashed{p}^\dagger/\Lambda^2}]$$

$$= \lim_{\Lambda \to \infty} \int \frac{d^4p}{(2\pi)^4 \Lambda^4} e^{-D_\mu D^\mu/\Lambda^2} \text{Tr}\left[\frac{1}{2}(\sigma^{\mu\nu} F_{\mu\nu})^2 - \frac{1}{2}(\bar{\sigma}^{\mu\nu} F_{\mu\nu})^2\right]$$

$$= \int \frac{\pi^2 p^2 dp^2}{(2\pi)^4} e^{-p^2} \cdot \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \text{Tr} F_{\mu\nu} F_{\rho\lambda} = \frac{1}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$  

Explicit form of Dirac operators and Pauli matrices:

$$\slashed{D}^\dagger \slashed{D} = D_\mu D^\mu + \frac{i}{4}(\sigma^\mu_\nu \bar{\sigma}^\nu_\nu - \sigma^\nu_\nu \bar{\sigma}^\mu_\mu)i[D_\mu, D_\nu] = D_\mu D^\mu - \sigma^{\mu\nu}_E F_{\mu\nu}$$

$$\slashed{D} \slashed{D}^\dagger = D_\mu D^\mu + \frac{i}{4}(\bar{\sigma}^\mu_\nu \sigma^\nu_\nu - \bar{\sigma}^\nu_\nu \sigma^\mu_\mu)i[D_\mu, D_\nu] = D_\mu D^\mu - \bar{\sigma}^{\mu\nu}_E F_{\mu\nu}$$

$$\text{Tr}[\sigma^{\mu\nu}_E \sigma^{\rho\lambda}_E - \bar{\sigma}^{\mu\nu}_E \bar{\sigma}^{\rho\lambda}_E] = \epsilon^{\mu\nu\rho\lambda}$$
Summary: the operator form of the chiral anomaly:

$$\partial_\mu j^\mu_A = A(x) = \frac{1}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$ 

The important point: current conservation at the classical level is violated by a quantum effect of a very specific form.
The Atiyah-Singer Index Theorem

Instructive alternative computation of $A(x)$:

$$g_n = \frac{1}{\lambda_n} \mathcal{D} f_n, \quad f_n = \frac{1}{\lambda_n} \mathcal{D}^\dagger g_n,$$

so contributions to $A(x)$ cancel in pairs except for $\lambda_n = 0$:

$$\int d^4x A(x) = \text{Tr} \int d^4x \sum_n (f_n^\dagger(x) f_n(x) - g_n^\dagger(x) g_n(x)) = n_\psi - n_{\bar{\psi}}.$$

The number of zero modes of $\mathcal{D}, \mathcal{D}^\dagger$: $n_\psi, n_{\bar{\psi}}$.

The Atiyah-Singer index theorem:

$$n_\psi - n_{\bar{\psi}} = \frac{1}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}.$$

Mathematical significance: a relation between a topological quantity (the difference in the number of zero-modes cannot change continuously) to an analytical quantity (an integral over a smooth function).

Towards the physical significance: there would be no chiral anomaly were it not for the zero modes of the Dirac operator (and those make the path integral vanish).
Instantons

The Euclidean action $S \sim \int d^4 x \text{Tr} F_{\mu\nu} F^{\mu\nu}$ of a gauge field configuration diverges at infinity, unless the gauge field is asymptotically pure gauge $A_\mu \rightarrow U(x) \partial_\mu U(x)\dagger$.

$\text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$ is a total derivative so its integral vanishes except for a surface term at the asymptotic Euclidean 3-sphere $S^3$. This surface term counts the wrapping number $\nu$ of the $U(x)$ around the asymptotic $S^3$.

Example: $G = SU(2) \sim S^3$ as a group, so the obvious embedding $G \rightarrow S^3$ has winding number $\nu = 1$.

Field configurations with winding number have non-trivial field strength in bulk. Their Euclidean action is finite but large for small coupling $S_E \sim 1/g^2$.

These configurations are localized in Euclidean space, including in time. They are called instantons.
The explicit integral over a winding configuration gives
\[ \frac{1}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \nu . \]

**Remark:** the winding number \( \nu \) is just the integral difference of zero modes \( n_\psi - n_{\bar{\psi}} \).

The integral above is in the fundamental representation (denoted \( \Box \)) where the gauge matrices \( U(x) \) are just the defining matrices.

The index \( T(\Box) = \frac{1}{2} \) in a general representation \( \mathbf{r} \) is
\[ \text{Tr}[T^a T^b] = T(\mathbf{r}) \delta^{ab} . \]

The anomaly for a theory with \( n(\mathbf{r}) \) chiral fermions in representation \( \mathbf{r} \):
\[ \int d^4x A(x) = \frac{1}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \sum_r n(\mathbf{r})2T(\mathbf{r}) . \]
The $\theta$-term

The gauge theory action generally includes the CP-violating term

$$S_{\text{CP}} = -\frac{\theta_{\text{YM}}}{16\pi^2} \int d^4 x \text{Tr} F^{\mu\nu} \tilde{F}_{\mu\nu}.$$  

The term is a total derivative but it does not vanish:

$$S_{\text{CP}} = -\theta_{\text{YM}} \nu.$$  

The path integral sums over all field configurations, with different instanton sectors weighed by the phase factor $e^{-i\theta_{\text{YM}} \nu}$.

Theories with different $\theta_{\text{YM}}$ are generally physically distinct. Specifically they have different spectrum.

Exception: shifting the phase $\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} + 2\pi \mathbb{Z}$ is inconsequential.
The chiral transformation $\psi \to e^{i\alpha} \psi$, $\bar{\psi} \to e^{-i\alpha} \bar{\psi}$ shifts the path integral as:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} \to \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS - i\alpha \int d^4xA(x)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS - i\frac{\alpha}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}}.$$

Equivalently:

$$\theta_{YM} \to \theta_{YM} - \alpha \sum_r n_r 2T(r).$$

Example: in pure SYM the gaugino is the only chiral fermion. A chiral rotation $\lambda^a \to e^{i\alpha} \lambda^a$ is equivalent to

$$\theta_{YM} \to \theta_{YM} - 2N\alpha.$$

because the index of the adjoint is $T(\text{Ad}) = N$.

Thus the chiral anomaly breaks the continuous $U(1)_R$ in the classical theory to the $Z_{2N}$ generated by $\alpha = \frac{k}{N}\pi$ (with $k = 0, \ldots, 2N - 1$).
The Euclidean action of instantons is bounded:

\[ 0 \leq \int d^4 x \text{Tr} (F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 = 2 \int d^4 x \left( F_{\mu\nu} F^{\mu\nu} \pm F_{\mu\nu} \tilde{F}^{\mu\nu} \right), \]

so

\[ \int d^4 x F_{\mu\nu} F^{\mu\nu} \leq \left| \int d^4 x F_{\mu\nu} \tilde{F}^{\mu\nu} \right| = 16\pi^2 \nu. \]

Instantons in fact saturate this bound.

Instanton contributions to the Euclidean path integral are suppressed by the characteristic nonperturbative factor:

\[ \left( e^{-\frac{8\pi^2}{g^2} + i\theta} \right)^\nu. \]

Such effects are qualitatively important because they are the leading contributions to some processes.

For example, we would like to break SUSY in a non-perturbative manner, where the breaking is heavily suppressed.
Triangle Diagrams

Alternative derivation of chiral anomaly: Triangle diagrams.

Chiral current (in the interacting theory):

\[ \langle j^\mu_A(x) \rangle_{\text{int}} = \langle \bar{\psi}(x) \gamma^\nu \gamma^5 \psi(x) \ e^{-ie \int d^4y \bar{\psi} A \psi} \rangle_0 \]

Expanding to second order, contract all fermions: find triangle diagram with current insertion at one angle and gauge field on two others:

The divergence of the chiral current vanishes formally:

\[ q_\mu \langle j^\mu_A(q) \rangle_{\text{int}} = 0 \quad (\text{formally}) . \]
The reason: explicit integrals for the two diagrams cancel upon linear shifts in the loop momentum variables.

**The catch:** shift of loop momentum is only justified if the regulated integral preserves the shift symmetry. *This is not possible* in any regularization scheme.

Explicit computations in several regularization schemes

$$\partial_\mu \langle j_A^\mu \rangle_{\text{int}} = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$
The Gauge Anomaly

Setting: a chiral gauge theory with matter in a chiral representation (the couplings are chiral).

Sample Lagrangian:

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^\mu D_\mu \psi_L + \psi_R^\dagger i\sigma \partial_\mu \psi_R , \quad D_\mu = \partial_\mu - igA^a_\mu t^a$$

Noether current of interest: the gauge current (the symmetry is not chiral):

$$j^{\mu a} = \psi_L \bar{\sigma}^\mu t^a \psi_L$$

Gauge anomaly: two triangle diagrams fail to cancel (but details quite different from the chiral anomaly).

The significance: the gauge current must be exactly conserved or else the decoupling of longitudinal gauge particles (and ghosts) fail.

Gauge theory is inconsistent at the quantum level if there is a gauge anomaly.
Evaluation:
1) explicit computation of triangle diagram.
or
2) topological argument (reducing path integral to zero modes).

Result at linear order (non-linear order not just field strength!):
\[ \partial_\mu \langle j^{\mu a} \rangle = -\frac{g^2}{16\pi^2} \epsilon^\mu_{\nu\alpha\beta} F^b_{\nu\mu} F^c_{\alpha\beta} A^{abc}, \]

where the anomaly coefficient:
\[ A^{abc} = \text{tr}[t^a, \{t^b, t^c\}] . \]

Consistency condition on gauge theory: the anomaly coefficient must vanish (when summed over all matter representations).
Remarkable fact: SM spectrum is such that all gauge anomalies vanish.

Example: anomaly of $U(1)_{\text{EM}} \times SU(2)^2_L$:

$$\text{tr}[Q, \{\tau^a, \tau^b\}] = \delta^{ab}\text{tr}[Q] = 3 \cdot \left(\frac{2}{3} - \frac{1}{3}\right) + (0 - 1) = 0.$$
Anomaly Matching

Important application of anomalies: ’t Hooft’s anomaly matching.

Setting: a UV theory with some **global symmetry** $G$. Anomaly matching gives information about the IR behavior of the theory.

Promote $G$ to a **gauge symmetry**, with some weak coupling $g$.

The anomaly coefficients $A_{UV}$ may fail to vanish. Add some spectator matter fields that only couple to $G$, designed such that $A_{UV} + A_S = 0$.

The gauge group $G$ is just a redundancy of the theory so it must remain in the IR theory.

The anomaly coefficients (including the contribution from the spectators) of the IR theory therefore must vanish $A_{IR} + A_S = 0$. Thus $A_{UV} = A_{IR}$.

The gauge bosons and the spectator fields decouple as $g \to 0$ but maintain the three point functions of the currents so that $A_{UV} = A_{IR}$ remains.