Lecture 10
Holomorphy: Chiral Field Th'y

Outline

- Discussion: the meaning of quantum field theory.
- Symmetries and selection rules in SUSY QFT.
- Nonrenormalization theorems in SUSY QFT.
- Example 1: superpotential in the WZ-model
- Example 2: mass thresholds.

Reading: Terning 8.1, 8.3

Effective Quantum Field Theory

Modern interpretation of QFT: effective theory that applies below some scale $\Lambda_{\rm cut-off}$. Details:

- We should focus on fields with mass below the scale $\Lambda_{\rm cut-off}$.
- The most important terms in the theory are the renormalizable ones (mass dimension ≤ 4) that can be expressed in terms of the light fields, consistent with symmetries.
- Other terms expressed in terms of the light fields must be allowed in principle; but these non-renormalizable terms (mass dimension > 4) are suppressed by positive powers of the large scale $\Lambda_{\rm cut-off}$.
- The "UV-theory" is a formal limit where $\Lambda_{\rm cut-off}$ is taken infinite. The non-renormalizable terms are taken to vanish identically.
- For conceptual clarity we often specify the UV-theory and then analyze the nature of the low energy physical theory. Many UV-theories have the same low energy theory.

The Physical Scale

- The cut-off scale $\Lambda_{\mathrm{cut-off}}$ is a physical scale.
- The coefficients of all terms in the Lagrangian (renormalizable and non-renormalizable alike) depend on scale — they "run" as the scale is lowered.
- If the scale is lowered below the mass of one or more fields, those must be integrated out. The resulting theory must be expressed in terms of excitations at lower energy.
- The relevant low energy excitations could be evident already in the high energy theory, or they could be qualitatively different.

One major motivation for studying SUSY QFT: these statements can be made quite explicit, while remaining very rich.

Scale Dependence in SUSY Theories

- Divergences in Feynman diagram computations isolate dependence of the low energy theory on the UV theory.
- This in turn determines the running of parameters in the low energy theory.
- Divergences in SUSY field theories are milder than in generic QFTs.
- Quadratic divergences cancel because of boson-fermion degeneracy in the spectrum.
- The coefficients in the superpotential do not run. This is an important example of a non-renormalization theorem.

Non-renormalization Theorems

- Traditional strategy: analyze diagrams (in superspace formalism). Find that coefficients in superpotential cannot be induced from virtual particles.
- Modern strategy: exploit symmetries.
- Key insight (Seiberg): holomorphy!

  Low energy superpotential $W_{\mathrm{eff}}(\phi)$ is a function of ϕ only (not ϕ^*), and and coupling constants are similarly holomorphic.
- Other symmetries: we must take proper advantage of selection rules from other symmetries.
- These are the key ingredients to analysis of (much) richer theories as well.

R-symmetry in Scalar Field Theory

Example: a single chiral superfield ϕ, with some superpotential $W_{\rm tree}(\phi)$.

R-symmetry: act as a phase factor on the superspace coordinate θ. By convention the R-charge $R[\theta] = 1$.

The superpotential term in the Lagrangian:

$$\mathcal{L}_{\rm int} = \int d^2\theta \, W(\phi) \ .$$

Grassmann integrals define $\int d^2\theta \; \theta^2$ as a pure number $(= 1)$, so $R[d^2\theta] = -2$. Thus: $U(1)_R$ is a symmetry if $R[W] = 2$.

Reminder: the scalar component of a chiral superfield is also denoted ϕ; the fermion component is ψ. By convention the R-charge of a chiral superfield is that of the scalar component. Then $R[\psi] = R[\phi] - 1$.

In component form (no superspace), these considerations rely heavily on the commutator

$$[R, Q_\alpha] = -Q_\alpha \ .$$

Selection rules: Spurion Fields

Example: WZ model

$$W_{\text{tree}}(\phi) = \tfrac{m}{2}\phi^2 + \tfrac{\lambda}{3}\phi^3 \ .$$

Assign $R[\phi] = 1$ so free theory ($\lambda = 0$) preserves R-symmetry.

Treat λ as a background field (spurion field) that has charge $R[\lambda] = -1$, so R-symmetry is preserved by interaction.

An ordinary (not "R-") symmetry does not act on θ, so the superpotential W is neutral under such symmetries.

Summarize symmetries in WZ-model:

	$U(1)$	$\times$	$U(1)_R$
ϕ	1		1
m	-2		0
λ	-3		-1

Remarks

Normalizations: the overall magnitude of $U(1)_R$ is set by the convention that $R[\theta]=1$. But other $U(1)$ charges can be normalized in any manner.

Ambiguity of R: the assignment of R-charges can be modified by multiples of other $U(1)$ charges, since those transformations leave θ invariant.

Anti-holomorphic sector: we pay attention to the holomorphic behavior alone. For the superpotential, there is also an anti-holomorhic term (so the Lagrangian is real), but it gives no additional information.

The Kinetic term is invariant under simultaneous $U(1)_R$ (or $U(1)$) transformations in the holomorphic and anti-holomorphic sectors, with conjugate phase factors for the two sectors:

$$\mathcal{L}_{\rm kin} = \int d^4\theta \ \Phi^\dagger \Phi \ .$$

(Superfield denoted for Φ for emphasis, the ϕ notation is more informal).

A Non-renormalization Theorem

Procedure: integrate out (in the Wilsonian sense) modes from Λ down to μ.

In detail: start from $W_{\rm tree}$ at scale Λ, expand in Fourier modes, and carry out the path integral over modes above μ (but below Λ since those modes are already averaged over). Assemble the result into a $W_{\rm eff}$ that depends on modes below μ only.

The claim: $W_{\rm eff}(\phi) = W_{\rm tree}(\phi)$ (non-renormalization).

Exploit Symmetries

Reminder: the global charges

	$U(1)$	$\times$	$U(1)_R$
ϕ	1		1
m	-2		0
λ	-3		-1

The superpotential must have R-charge 2 and vanishing $U(1)$ charge. For example $W_{\text{mass}} = \frac{1}{2} m\phi^2$ has those charges.

Another ingredient: the combination $\lambda\phi/m$ is neutral under both $U(1)$'s. General form of effective superpotential:

$$W_{\text{eff}} = m\phi^2 \, h\left(\frac{\lambda\phi}{m}\right)$$

where h is an unknown holomorphic function.

Exploit Holomorphy and Limits

Power series:

$$W_{\text{eff}} = m\phi^2\, h\left(\frac{\lambda\phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \phi^{n+2} \ ,$$

Weak coupling limit: $\lambda \to 0$ must give just the mass term since in this case there are no interactions. So $n \geq 0$.

The massless limit: $m \to 0$ is a sensible limit. For example it could be taken with some small finite λ and then we expect a nice perturbation series in λ. The general form restricts $n \leq 1$.

The upshot: only $n = 0$ and $n = 1$ are allowed! Renaming the coefficients, we find

$$W_{\text{eff}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3 = W_{\text{tree}}$$

The superpotential is not renormalized!

Discussion

The conclusion: holomorphic couplings in the superpotential as not renormalized.

The quantum effects in chiral field theory: all in the wave function renormalization factor Z.

Equivalently:

$$\mathcal{L}_{\rm kin} = \int d^4\theta \; \Phi^\dagger \Phi \to \int d^4\theta \; K(\Phi^\dagger, \Phi) \; .$$

The challenge:
Symmetries does not constrain the Kähler potential significantly!

Status: this challenge cannot be overcome generally. Yet, many features of low energy effective theory can be established using holomorphy and other techniques.

Example: Integrating out

Theory: a model with two different chiral superfields

$$W_{\rm tree} = \tfrac{1}{2} M \phi_H^2 + \tfrac{\lambda}{2} \phi_H \phi^2$$

Three global $U(1)$ symmetries:

	$U(1)_A$	$U(1)_B$	$U(1)_R$
ϕ_H	1	0	1
ϕ	0	1	$\frac{1}{2}$
M	-2	0	0
λ	-1	-2	0

Note: $U(1)_A$ and $U(1)_B$ are spurious symmetries for M, $\lambda \neq 0$.

Integrating out

If we want an effective scale $\mu < M$, we must integrate out the heavy ϕ_H with mass M.

General holomorphic form of the effective superpotential

$$\phi^j M^k \lambda^p$$

Symmetry considerations:

- Preserve R-symmetry: $j = 4$
- Preserve $U(1)_B$: $p = \frac{1}{2}j = 2$.
- Preserve $U(1)_A$: $k = -\frac{1}{2}p = -1$.
- A limit: compare with tree-level perturbation theory to determine overall numerical factor.

Conclusion: the quantum corrected effective superpotential is:

$$W_{\text{eff}} = -\frac{\lambda^2 \phi^4}{8M}$$

Perspective: consider the classical equation of motion:

$$\frac{\partial W}{\partial \phi_H} = M\phi_H + \frac{\lambda}{2}\phi^2 = 0$$

Solve this equation for ϕ_H, and insert the result back into the original superpotential W_{tree}. We recover W_{eff}.

Interpretation: at low energy we expect the heavy field ϕ_H to take its classical value, with interacting quantum fluctuations around that value to be integrated over in the path integral. The result shows that these quantum fluctuations cancel.

Another Example

A more complicated theory:

$$W_{\text{tree}} = \tfrac{1}{2} M \phi_H^2 + \tfrac{\lambda}{2} \phi_H \phi^2 + \tfrac{y}{6} \phi_H^3$$

Classical equation of motion for ϕ_H:

$$\phi_H = -\frac{M}{y}\left(1 - \sqrt{1 - \frac{\lambda y \phi^2}{M^2}}\right)$$

Took lower branch so $\phi_H \to -\frac{\lambda\phi}{2M}$ as $y \to 0$, to recover the previous example. (The upper branch is in fact classically unstable).

Insert classical ϕ_H in superpotential:

$$W_{\text{eff}} = \frac{M^3}{3y^2}\left(1 - \frac{3\lambda y \phi^2}{2M^2} - \left(1 - \frac{\lambda y \phi^2}{M^2}\right)\sqrt{1 - \frac{\lambda y \phi^2}{M^2}}\right)$$

Holomorphy

Using holomorphy: assign the coupling y charges $(-3, 0, -1)$ under $U(1)_A \times U(1)_B \times U(1)_R$. Then the symmetries for $y = 0$ at all preserved.

The combination $\frac{\lambda y \phi^2}{M^2}$ of the low energy variables is neutral under all charges. So general form of superpotential:

$$W_{\text{eff}} = \frac{M^3}{y^2} f\left(\frac{\lambda y \phi^2}{M^2}\right)$$

for some function f.

The classical computation indeed found W_{eff} of this form, with a specific function f.

Classical computation exact in limit where couplings $\lambda, y \to 0$. Can take limit with $M \to 0$ such that W_{eff} retains its functional form. Therefore the classical computation is exact.

Singularities

There are singularities in the low energy $W_{\rm eff}$.

Interpretation: these are points in the parameter space and the space of ϕ VEVs where the low energy theory breaks down.

Explicit verification: the mass of ϕ_H can be found by calculating

$$\frac{\partial^2 W}{\partial \phi_H^2} = M + y\phi_H$$

and substituting in the solution for ϕ_H:

$$\frac{\partial^2 W}{\partial \phi_H^2} = M\sqrt{1 - \frac{\lambda y \phi^2}{M^2}}$$

Conclusion: for ϕ VEVs such that ϕ_H becomes massless, it is not justified to integrate it out. The singularity in the effective superpotential signals this error.

Summary

- Modern effective QFT: the low energy theory after heavy particles have been integrated out.
- A non-renormalization theorem: the Wilsonian superpotential does not receive quantum corrections.
- Mass thresholds: integrate out by solving equation of motion for heavy d.o.f. (careful if the "heavy" d.o.f. is in fact light).