PHY 513: HW 6 (due thu. oct 22, 2009)

Before turning to the problems, we take this opportunity to summarize our conventions. The notation used here (and in lecture) differs from **PS** by a little bit and is closer to what is standard in other books and the literature.

In our notation parity acts as

$$\mathcal{P}\psi(t,\vec{x})\mathcal{P}^\dagger = \eta_a \gamma^0 \psi(t,-\vec{x})$$

Note that there is a $\mathcal{P}^\dagger$ on the right. The notation makes the unitary nature of the transformation manifest (and is at any rate more standard).

In **PS** there is a $\mathcal{P}$ on the right. From a practical point of view this is a minor change since $\mathcal{P}^\dagger$ and $\mathcal{P}$ differ only by a phase which in fact can be chosen to unity. But having unitarity manifest is better conceptually, and also leads to fewer mistakes.

Our convention for charge conjugation $\mathcal{C}$ is similarly

$$\mathcal{C}\psi(t,\vec{x})\mathcal{C}^\dagger = -i(\bar{\psi}(t,\vec{x})\gamma^0\gamma^2)^T$$

and time reversal acts on spinors as

$$\mathcal{T}\psi(t,\vec{x})\mathcal{T}^\dagger = \gamma^1\gamma^3\psi(-t,\vec{x})$$

The $\mathcal{T}$ operator is anti-unitary *i.e.* $\mathcal{T}c\mathcal{T}^\dagger = c^*$ on complex numbers (and Dirac matrices).

The operators $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{T}$ are identical to the operators $U(P)$, $U(C)$, and $U(T)$ used in the lectures. The latter notation is perhaps better for hand-writing.

In the literature one also often sees notations such as $P = \gamma^0$, $C = -i\gamma^0\gamma^2$, $T = \gamma^1\gamma^3$ for the 4×4-matrices that transform the spinors on the right hand sides of the formulae above. We will not use these abbreviations in this course.

1 P, C, and T

The transformation properties of fermion bilinears under the discrete tranformations C, P, T are summarized in the table on **PS** page 71.

a) Verify the *parity transformation* properties of the pseudo-vector $A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. and the tensor $T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$.

The pseudo-vector is sometimes referred to as an axial-vector. This accounts for the notation A^μ (which should not be confused with the electromagnetic gauge potential).

b) Verify the transformations under *charge conjugation* of the vector $V^\mu = \bar{\psi}\gamma^\mu\psi$ and the pseudo-vector $A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$.

c) Verify the transformations under *time-reversal* of the pseudo-scalar $P = i\bar{\psi}\gamma^5\psi$ and the vector $V^\mu = \bar{\psi}\gamma^\mu\psi$.

2 Discrete Symmetry in Nature

a) Determine the transformation properties of the bilinear operators $V^\mu = \bar{\psi}\gamma^\mu\psi$ and $A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ under CP (*ie* the P-operation followed by the C-operation).

b) According to QED the electromagnetic interaction takes the form

$$\mathcal{L}_{\rm QED} = J_\mu(x) A^\mu_{\rm em}(x)$$

where J_μ is the current and $A^\mu_{\rm em}$ is the electro-magnetic gauge potential. For a single fermion (such as the electron) the current takes the form $J_\mu = eV_\mu$ where $V^\mu = \bar{\psi}\gamma^\mu\psi$.

The electromagnetic potential $A^\mu_{\rm em}$ transforms as

$$\begin{aligned} \mathcal{C} A^\mu_{\rm em}(t,\vec{x})\mathcal{C}^\dagger &= -A^\mu_{\rm em}(t,\vec{x}) \\ \mathcal{P} A^\mu_{\rm em}(t,\vec{x})\mathcal{P}^\dagger &= A_{\mu\ \rm em}(t,-\vec{x}) \end{aligned}$$

under charge-conjugation and parity. Is the QED interaction invariant under P? Under C? Under CP?

c) The Lagrangean for the weak interactions at low energies takes the form

$$\mathcal{L}_{\rm weak} = \frac{G_F}{\sqrt{2}}(V_\mu - A_\mu)(V^\mu - A^\mu)$$

where G_F is a (dimensionful) coupling constant. Is this Lagrangian invariant under P? Under C? Under CP?

3 CPT

Define the product of the 3 discrete symmetry transformations as $\Theta \equiv \mathcal{CPT}$. Show that under Θ, the Dirac field transforms as

$$\Theta\psi(x)\Theta^\dagger = \gamma^5\psi^*(-x),$$

where $\psi(x)^* \equiv (\psi(x)^\dagger)^T$. Take the phase associated with parity $\eta_a = 1$.

4 Representations of the Dirac Matrices

The Dirac Matrices are defined as a set of 4×4 matrices that satisfy the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. For many purposes this algebra is all that is needed. Whenever a specific representation has been needed **PS** chooses the matrices

$$\gamma_W^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \; \gamma_W^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{1}$$

In this representation all the γ's are off-diagonal when written in terms of 2×2 sub-matrices, and so all the Lorentz generators $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are diagonal. This makes the interpretation of the 4-spinor in terms of chiral components — the Weyl 2-spinors — particularly transparent. This representation is therefore known as the chiral representation, or as the Weyl representation. In this problem we use the subscript W to emphasize this aspect.

a) There are many other representations of the Dirac algebra. Indeed consider new matrices

$$\gamma^\mu = U\gamma_W^\mu U^\dagger$$

where U is some unitary 4×4 matrix. Show that these γ^μ satisfy the Dirac algebra as well. It can be shown that *all* possible realizations of the Dirac algebra are unitarily equivalent, *i.e.* all representations can be constructed this way.

b) A frequent alternative to the Weyl representation corresponds to

$$U_D = \frac{1}{\sqrt{2}}(1 - \gamma_W^5\gamma_W^0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

This defines a representation known as the Dirac representation (or, misleadingly, the standard representation). Show that U_D is unitary and find the Dirac matrices in the Dirac representation.

c) Write the positive frequency solutions to the Dirac equation as

$$\psi(x) = u(p)e^{-ipx}$$

as usual. Verify that the solution in the DIrac representation is

$$u_D^s(p) = \frac{1}{\sqrt{E+m}} \begin{pmatrix} (E+m)\xi^s \\ \vec{\sigma}\cdot\vec{p}\xi^s \end{pmatrix} \tag{2}$$

d) Show that the solution above is normalized in the conventional way, so that $\bar{u}u = 2m$.

Comments: the spinor written in the *Dirac representation* (2) has vanishing lower components in the restframe and, at more general low momenta, the two upper component dominates the lower two. The two "large" upper components can be interpreted as the usual two component Schrödinger wave function for a particle of spin-$\frac{1}{2}$. The Dirac representation is therefore useful for low energy processes, *e.g.* for computing relativistic corrections to non-relativistic quantum mechanics.

The *Weyl representation* is the better one for relativistic processes and — therefore — also for understanding how the Lorentz group acts on the spinors. Most particle physicists prefer this representation and, because the group theory is easier to analyze, it is best for pedagogical purposes.

There is a third representation that is in relatively common use: the *Majorana representation*. In this representation all the Dirac-matrices are real. It is convenient in situations where particles and anti-particles are identified.