

PHY 513: HW 5 (due tue. oct 13, 2009)

1 Algebra of Dirac Matrices

Verify

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu.$$

using the fundamental anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and the reordering relation

$$[A, BC] = \{A, B\}C - B\{A, C\}.$$

The $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$ are the representation matrices for the vector representation, given in **PS** (3.18).

The algebraic relation proven in this exercise can easily be rewritten as **PS** (3.29) which is a central property of the Dirac γ^μ -matrices.

2 Contraction Identities

Let k, p, q represent three arbitrary four-vectors. Using the anti-commutation relations for the Dirac matrices, prove the identities:

- $\gamma_\mu \gamma^\mu = 4.$
- $\gamma_\mu \not{k} \gamma^\mu = -2\not{k}.$
- $\gamma_\mu \not{p} \not{q} \gamma^\mu = 4p \cdot q.$
- $\gamma_\mu \not{k} \not{p} \not{q} \gamma^\mu = -2\not{q} \not{p} \not{k}.$

These identities are given in **PS** as (5.8)-(5.9). They will be essential for evaluating Feynman diagrams involving spin- $\frac{1}{2}$ particles.

3 The Gordon Identity

Solve **PS** 3.2. You need the definition

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

which is given in the beginning of **PS** sec. 3.4.

4 γ^5 and Completeness

Consider the matrix $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

a) Show that γ^5 anticommutes with each of the γ^μ matrices,

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

c) Show that

$$\gamma^5 = -\frac{i}{24}\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$$

and

$$\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$$

We use the sign convention: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$.

The significance of the result under c) is the following. The 16 matrices $1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$ are a basis for all 4×4 matrices so it is possible to express any 4×4 matrix as a linear combination of these. How would one do this concretely? Start with a product of any number of γ -matrices. Whenever there is two adjacent γ -matrices, rewrite them as a symmetric combination (which the anti-commutation relations shows is proportional to $g^{\mu\nu}$ times the identity matrix) and an anti-symmetric combination. Continue doing this until all terms are fully antisymmetric in all indices, or written in terms of the metric $g^{\mu\nu}$. Anti-symmetric combinations of more than four γ -matrices vanish because there are just four γ -matrices. Now, the anti-symmetric combination of four γ -matrices were reduced to γ^5 in c) and computations similar to c) show that combinations of a smaller set of γ -matrices can be rewritten as

$$\begin{aligned}\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} &= i\epsilon^{\lambda\mu\nu\kappa}\gamma_\kappa\gamma^5 \\ \gamma_5\sigma_{\mu\nu} &= -\frac{i}{2}\epsilon_{\mu\nu\lambda\kappa}\sigma^{\lambda\kappa}\end{aligned}\tag{1}$$

These manipulations thus give a constructive way to reduce complicated products of γ -matrices to a linear combination of the "irreducible" matrices $1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$.

5 Helicity Spinors

The eigenstates of helicity for spin- $\frac{1}{2}$ generally take the form

$$u(p) = \begin{pmatrix} \sqrt{E - \vec{\sigma} \cdot \vec{p}} \xi \\ \sqrt{E + \vec{\sigma} \cdot \vec{p}} \xi \end{pmatrix}, \quad (2)$$

where ξ are some two-spinors. There are several different useful ways to specify the two-spinors. The helicity four-spinors u^\pm correspond to the choice where the two spinors have fixed helicity, ie. they have spin $\pm\frac{1}{2}$ along the direction of motion.

Determine the two-component helicity spinors $\xi_\pm(\hat{\mathbf{p}})$ by solving the eigenvalue equation for helicity

$$\left(\hat{\mathbf{p}} \cdot \frac{\vec{\sigma}}{2} \right) \xi_\pm(\hat{\mathbf{p}}) = \pm \frac{1}{2} \xi_\pm(\hat{\mathbf{p}}), \quad (3)$$

where $\hat{\mathbf{p}}$ is the unit vector in the direction of $\vec{\mathbf{p}}$. Assume that the spin- $\frac{1}{2}$ particle has mass m and momentum

$$\vec{\mathbf{p}} = p(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

where $p = |\vec{p}|$. Normalize the spinors ξ such that $\xi^\dagger \xi = 1$ and choose their overall phases so that, for a particle moving in the $+\hat{z}$ -direction, they reduce to the usual spin-up/spin-down forms

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (4)$$