PHY 513: HW 4 (due tue. oct 6, 2009)

1 Coherent States

A *coherent state* takes the form

$$|\{\eta_k\}\rangle \equiv \mathcal{N}\exp\left\{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}\right\}|0\rangle, \tag{1}$$

The η_k are some complex numbers which characterize the state and the $|0\rangle$ is the usual "vacuum" state which is annihilated by all the a_k, *i.e.* $a_k|0\rangle = 0$. The $\mathcal{N}$ is a normalization factor. As in **PS** the normalization is taken to be

$$\left[a_k, a_\ell^\dagger\right] = (2\pi)^3\delta(\vec{k}-\vec{\ell})$$

a) Show that

$$a_p|\{\eta_k\}\rangle = \frac{\eta_p}{\sqrt{2E_p}}|\{\eta_k\}\rangle.$$

One way to do this is to first show

$$\left[a_p, \exp\left\{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}\right\}\right] = \frac{\eta_p}{\sqrt{2E_p}}\exp\left\{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}\right\}$$

using a formula of the schematic form

$$\left[a_p, (a_k^\dagger)^m\right] = m(a_k^\dagger)^{m-1}(2\pi)^3\delta(\vec{p}-\vec{k})$$

b) Determine $\mathcal{N}$ such that the norm of the state is equal to 1.
c) Compute the expectation value

$$\overline{\phi(x)} \equiv \langle\{\eta_k\}|\phi(x)|\{\eta_k\}\rangle \tag{2}$$

d) Compute the average particle number in the state:

$$\overline{N} = \langle\{\eta_k\}|N|\{\eta_k\}\rangle$$

where the number operator is defined as $N \equiv \int \frac{d^3k}{(2\pi)^3}a_k^\dagger a_k$.
e) Calculate the mean square dispersion in particle number in this state, viz.,

$$\langle\Delta N^2\rangle \equiv \langle\{\eta_k\}|(N-\overline{N})^2|\{\eta_k\}\rangle,$$

2 Representations of $SO(1,3)$

Solve Problem PS3.1 a+b.

3 Discrete Symmetries

A general *Lie algebra* takes the form

$$[T_a, T_b] = i f_{abc} T_c \tag{3}$$

where a is some index $a = 1, \cdots, \dim g$ and the *structure constants* f_{abc} are real numbers that characterize the algebra. The T_a are abstract generators of the algebra. A *representation* of the algebra is a realization of the algebra in terms of some matrices T_a. Two different representations are *equivalent* if the corresponding matrices T_a and T'_a are related through $T'_a = U T_a U^\dagger$ for some unitary U.

The most elementary example of a Lie algebra is $g = SU(2)$, the infinitesimal form of the rotation group. In this case $\dim g = 3$ so that $a = 1, 2, 3$ and $f_{abc} = \epsilon_{abc}$. Irreducible representations are indexed by $j = 0, 1/2, 1, \cdots$; they have dimension $2j + 1$.
a) Consider a representation of some Lie group given by matrices T_a. Show that the matrices $-T_a^*$ also form a representation of the algebra.
b) Similarly, show that $T_a^\dagger$ also forms a representation.
c) For the case of $SU(2)$, we know that there is a two-dimensional representation realized by the Pauli matrices, $T_a \equiv \frac{\sigma_a}{2}$. Show explicitly that in this case the representation $-T_a^*$ is equivalent to T_a by establishing that

$$-T_a^* = S T_a S^{-1} \tag{4}$$

for some S. (Hint: look for S among the Pauli-matrices).
(d) Consider the chiral spinor representations of $SO(1,3)$, denoted as $L = (\frac{1}{2}, 0)$ and $R = (0, \frac{1}{2})$. Show that $L^* = R$ *i.e.* if the generators of L are denoted T_a then $-T_a^*$ are the generators of R. Why are L and R inequivalent representations?