PHY 513: HW 2 (due Tue 9/22/09)

1 Scalar Field Equations

For a real scalar field ϕ the Lagrangean density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \tag{1}$$

a) Find the Euler-Lagrange equation.
b) Determine the canonical momenta and the Hamiltonian.
c) Repeat a) and b) for a complex scalar field with Lagrangean

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - V(\phi^*\phi) \tag{2}$$

d) Show that the Lagrangean in c) is invariant under phase rotations $\phi \to e^{i\alpha}\phi$ and similarly for ϕ^*; find the corresponding Noether current.

2 EM-tensor for Electromagnetism

Problem 2.1 in **PS**.

3 Canonical Commutation Relations

Define

$$(f, g) \equiv i \int d^3x f^*(x) \overleftrightarrow{\partial_0} g(x) \tag{3}$$

where

$$f^*(x) \overleftrightarrow{\partial_0} g(x) \equiv f^*(x)\partial_0 g(x) - \partial_0 f^*(x)\ g(x) \tag{4}$$

The sesquilinear functional (f, g) is an inner product on the space of solutions to the Klein-Gordon equation with a given mass.
a) The inner product involves a 3-dimensional integration at fixed time. Show that the entire expression is independent of this time parameter. To do this, simply take the time-derivative of the whole expression and show that the result vanishes due to the Klein-Gordon equation.
b) Show that

$$a_{\vec{k}} = (f_k(x), \phi(x)) \tag{5}$$

where $f_k(x) = \frac{1}{\sqrt{2E_k}} e^{-ik\cdot x}$ and $\phi(x)$ is a real scalar field. This relation inverts the canonical expansions of ϕ, π in terms of $a, a^\dagger$ by writing a in terms of ϕ and $\dot{\phi} = \pi$.

c) Show that the canonical commutation relations for the field ϕ implies that

$$[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \tag{6}$$

Strategy: insert the result from b) and its complex conjugate. Whenever time-derivatives occur, trade them for canonical momenta. Then use the equal-time commutators for the field to simplify the resulting expression.

In **PS** it was shown that eq.6 implies the correct equal-time commutator of ϕ, π. The computation in this problem verifies the equivalence in the opposite direction.