

## PHY 523: HW 9 (due tue 3/30/09).

### 1 Asymptotic symmetry

This is an adaptation of **PS** problem 12.3.

Consider the quantum field theory with Lagrangean

$$\mathcal{L} = \frac{1}{2}((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2) - \frac{\lambda}{4!}(\phi_1^4 + \phi_2^4) - \frac{2\rho}{4!}(\phi_1^2\phi_2^2)$$

- a) Determine the vertices in this theory and the associated vertex factors.
- b) Show that the  $\beta$ -functions of the two couplings are

$$\begin{aligned}\beta_\lambda &= \frac{3}{16\pi^2} [\lambda^2 + (\rho/3)^2] \\ \beta_\rho &= \frac{1}{8\pi^2} [\lambda + 2(\rho/3)] \rho\end{aligned}$$

- c) Compute the  $\beta$ -function corresponding to the *ratio* of couplings  $\lambda/\rho$ .
- d) Show that if  $\lambda/\rho > 1/3$  the renormalization group flow is towards  $\lambda/\rho = 1$ . In particular, show that  $\lambda/\rho = 1$  is an attractive fixed point.
- e) Show that when  $\lambda = \rho$  the symmetry between the fields  $\phi_1$  and  $\phi_2$  is enhanced to a continuous  $O(2)$  symmetry interchanging the two fields. This significance of the result in d) is therefore that, at large distances, the theory will effectively have a  $O(2)$  symmetry, even if the Lagrangean did not.
- f) Draw a diagram that shows the flow of the couplings  $\lambda$  and  $\lambda/\rho$ .

### 2 Asymptotic Freedom

In non-abelian gauge theories the coupling constant  $g$  is "asymptotically free". This means that its  $\beta(g)$  is negative for small  $g$ . To one loop order

$$M \frac{d}{M} g \Big|_{g_0, \epsilon} \equiv \beta(g) = -\frac{\beta_1 g^3}{16\pi^2}$$

where  $\beta_1 > 0$  is some number of  $\mathcal{O}(1)$ . Similarly, the anomalous dimension of the field is of the form

$$\gamma(g) = \frac{\gamma_1 g^3}{16\pi^2}$$

where  $\gamma_1 > 0$ . The quantum is a vector in these theories but, in this exercise, we will suppress the corresponding vector indices.

The renormalized correlation functions satisfy the Callan-Symanzik equations which, for the amputated correlators, take the form

$$\left[ M \frac{d}{dM} + \beta(g) \frac{\partial}{\partial g} - n\gamma(g) \right] \Gamma_R^{(n)}(p_i/M, g) = 0$$

Taking all the momenta equal for simplicity, the solutions of the Callan-Symanzik equations take the form

$$\Gamma_R^{(n)}(p/M, g) = \Gamma^{(n)}(1, \bar{g}(p)) \cdot \exp\left(-4 \int_M^p d \log(p'/M) \gamma(\bar{g}(p'; g))\right)$$

This formula is analogous to **PS** 12.79, except that here it is written for the amputated correlation function.

- a) Compute the running coupling  $\bar{g}(p)$ .
- b) Compute the leading behavior of the running coupling as  $p/M \rightarrow \infty$ .
- c) Suppose the four point function to the lowest order in perturbation theory is  $\Gamma_R^{(4)} = g^2$ . Use the Callan-Symanzik equation to determine the dependence of the four-point vertex on momentum as  $p/M \rightarrow \infty$ .