1 Vacuum Conditions

Consider the WZ-model with the superpotential

\[ W = m\phi_2\phi_3 + \frac{1}{2}y\phi_1\phi_3^2 \]

1) Calculate the scalar potential.

2) Show that the SUSY vacua are given by

\[ \langle \phi_1 \rangle = v, \quad \langle \phi_2 \rangle = 0, \quad \langle \phi_3 \rangle = 0, \]

for arbitrary \( v \).

3) Expanding around the vacua \( \phi_1 = v \) and writing the scalar potential out to quadratic order, find the scalar mass squared matrix \( M_{\phi}^2 \).

4) Find the fermion mass matrix \( M_{\psi} \) and verify that \( M_{\psi} M_{\psi}^\dagger = M_{\phi}^2 \).

2 WZ-Model with Nonrenormalizable Terms

Consider a Wess-Zumino model with the superpotential

\[ W = \frac{1}{3}y\phi^3 + \frac{\lambda}{4M}\phi^4 \]

1) What are the off-shell SUSY transformations of the scalar \( \phi \) and its superpartner fermion \( \psi \), expressed only in terms of \( \phi \) and \( \psi \)?

2) For the superpotential given above, what is the corresponding Lagrangian in terms of \( \phi \) and \( \psi \)?

3) Schematically (include the parametric dependence on the couplings but not numerical coefficients), what are the Feynman rules for the cubic and quartic interactions?
3 SUSY invariance of SYM

Check that
\[
S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + i\lambda^i\bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a \right)
\]
is invariant under the SUSY transformations

Hints:

- After writing the SUSY variation it is extremely helpful to go to gauge where at the point of interest \( x_0^\mu \) the gauge field vanishes \( A^a_\nu(x_0) = 0 \).

- You will need to use the Bianchi identity
  \[
  \epsilon^\mu\nu\alpha\beta (D_\nu F_{\alpha\beta})^a = 0
  \]

4 SUSY Commutators

Check that the commutator of two SUSY transformations close in super Yang-Mills theory:
\[
(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})X^a = -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger)D_\mu X^a
\]
for \( X^a = F^a_{\mu\nu}, \lambda^a, \lambda^i, D^a \).

You need the identities:
\[
\begin{align*}
\xi \sigma^\mu \sigma^\nu \chi &= \chi \sigma^\nu \sigma^\mu \xi = (\chi^\dagger \sigma^\nu \sigma^\mu \chi^\dagger)^* = (\xi^\dagger \sigma^\mu \sigma^\nu \chi^\dagger)^*, \\
\sigma^\mu \sigma^\nu \sigma^\rho &= -\eta^{\mu
u\rho} + \eta^{\nu\mu\rho} + \eta^{\mu\rho\nu} + i\epsilon^{\mu\nu\rho\kappa} \sigma_\kappa \\
\sigma^\mu_{\alpha\alpha} \bar{\sigma}^\beta_{\beta} &= 2\delta^\beta_\alpha \delta^\beta_\alpha
\end{align*}
\]

5 SUSY invariance of Gauge Theory

Verify that the SUSY transformation of the gauge field \( A^a_\mu \) in
\[
\mathcal{L}_{\psi \text{ gauge int.}} = -g\psi^\dagger \bar{\sigma}^\mu A^a_\mu T^a \psi
\]
cancels in
\[
\mathcal{L}_{\text{Yukawa}} = -\sqrt{2}g[\phi^* T^a \psi] \lambda^a + \lambda^i a (\psi^\dagger T^a \phi]
\]
You will need to use the generalized Pauli identity (A.27).