Wandering domains for polynomial maps of $\mathbb{C}^2$

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No wandering domains

Theorem (Sullivan)

Every Fatou component of a rational map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ *is eventually periodic.*

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What about higher dimension ?

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Quiz

What is Hamal birth date ?

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Every Fatou component of a rational map f $: \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ *is eventually periodic.*

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What about higher dimension ?

Quiz

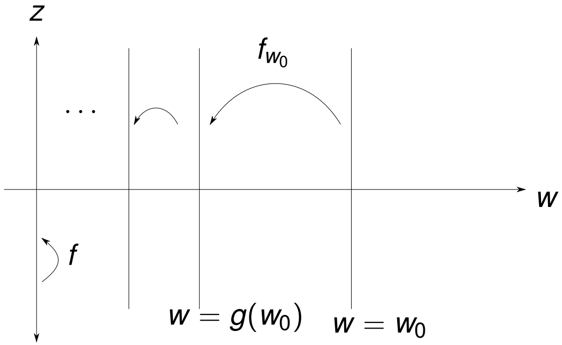
What is Hamal birth date ?
He has two of them : october 6 and october 7, 1945.

Skew products

Misha Lyubich realized that one could find wandering domains for polynomial skew products of the form

$$(w,z) \mapsto \big(g(w), f(z) + b(w)\big) \quad \text{with}$$

$$g(w) = w + \mathcal{O}(w^2), \quad f(z) = z + \mathcal{O}(z^2) \quad \text{and} \quad b(w) = \mathcal{O}(w).$$



Existence of wandering domains

Theorem (Astorg-Buff-Dujardin-Peters-Raissy)

If $a < 1$ is sufficiently close to 1, the skew product $P : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$P(w, z) := \left(w - w^2, z + z^2 + az^3 + \frac{\pi^2}{4} w \right)$$

has a wandering domain.

Experimentally, $a = 0.95$ works.

Quiz

Who was in the room in Hilleröd when Hamal said :
- Listen Adam. I do not understand what you are saying. Who in this room could understand what you are saying ?

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Computer illustration.

Lavaurs maps

- $f(z) = z + z^2 + \mathcal{O}(z^3)$ has a parabolic basin $\mathcal{B}_f$.
- $\mathcal{L}_\sigma : \mathcal{B}_f \to \mathbb{C}$ is the *Lavaurs map* with phase $\sigma \in \mathbb{C}$.
- $f_\varepsilon(z) = f(z) + \varepsilon^2$.

Theorem (Lavaurs)

Assume $T_n \to +\infty$ *and* $\varepsilon_n \to 0$ *are such that*

$$T_n - \frac{\pi}{\varepsilon_n} \to \sigma \in \mathbb{C}.$$

Then

$$f_{\varepsilon_n}^{\circ T_n} \to \mathcal{L}_\sigma$$

locally uniformly of the parabolic basin $\mathcal{B}_f$.

Existence of wandering domains

- $g(w) = w - w^2 + \mathrm{O}(w^3)$,
- $f(z) = z + z^2 + \mathrm{O}(z^3)$,
- $P(w, z) = \left(g(w), f(z) + \frac{\pi^2}{4} w\right)$.

Proposition

If the Lavaurs map $\mathcal{L}_0 : \mathcal{B}_f \to \mathbb{C}$ has an attracting fixed point, then the skew product P admits a wandering Fatou component.

Proposition

If $f(z) = z + z^2 + az^3$ with $a < 1$ sufficiently close to 1, then $\mathcal{L}_0$ has an attracting fixed point.

Key Lemma

- $g(w) = w - w^2 + \mathrm{O}(w^3)$ with a parabolic basin $\mathcal{B}_g$,
- $f(z) = z + z^2 + \mathrm{O}(z^3)$ with a parabolic basin $\mathcal{B}_f$,
- $P(w, z) = \left(g(w), f(z) + \dfrac{\pi^2}{4} w\right)$.

Lemma

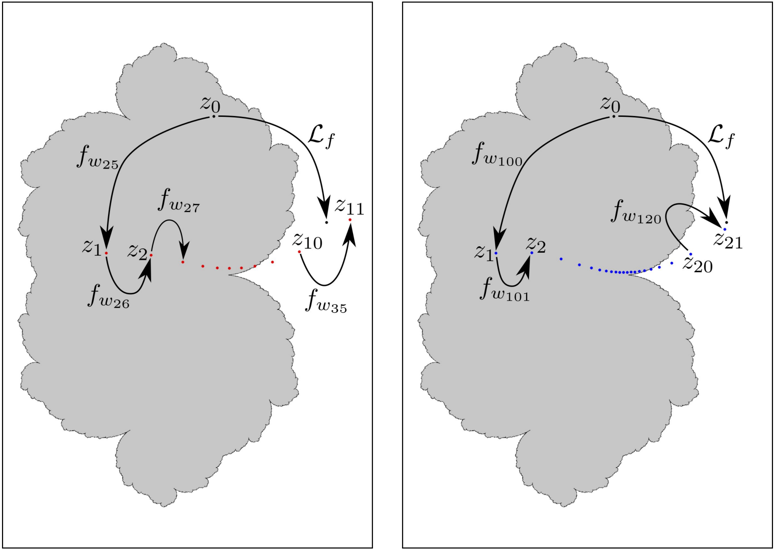
The sequence of maps

$$\mathbb{C}^2 \ni (w, z) \mapsto P^{\circ 2n+1}\left(g^{\circ n^2}(w), z\right) \in \mathbb{C}^2$$

converges locally uniformly on $\mathcal{B}_g \times \mathcal{B}_f$ to

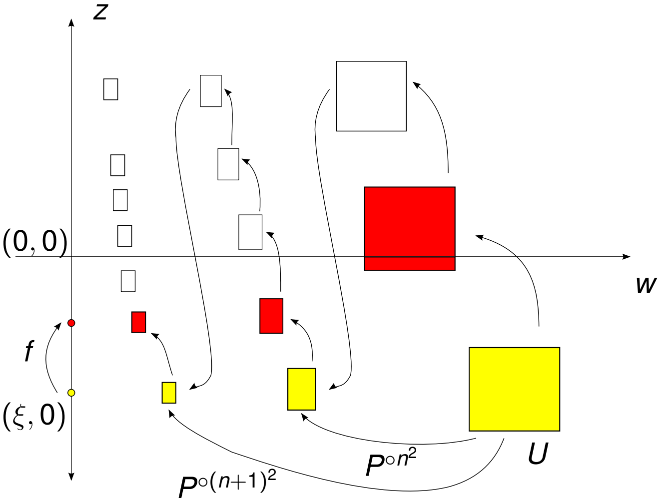
$$\mathcal{B}_g \times \mathcal{B}_f \ni (w, z) \mapsto \left(0, \mathcal{L}_0(z)\right) \in \{0\} \times \mathbb{C}.$$

Key Lemma



z_0
$\mathcal{L}_f$
$f_{w_{25}}$
$f_{w_{27}}$
z_{11}
z_1
z_2
z_{10}
$f_{w_{26}}$
$f_{w_{35}}$
z_0
$\mathcal{L}_f$
$f_{w_{100}}$
$f_{w_{120}}$
z_1
z_2
z_{21}
z_{20}
$f_{w_{101}}$

Existence of wandering domains



Back to the key Lemma

- Classical situation : $f_\varepsilon(z) = f(z) + \varepsilon^2$.
  - In appropriate coordinates, f_ε is close to a rotation by angle 2ε radians.
  - If $\varepsilon \simeq \pi/(2n)$, iterating $2n$ times, we make a full turn of 2π radians and get close to the Lavaurs map $\mathcal{L}_0$.
- Skew product. We consider the composition of $2n+1$ transformations $f_{\varepsilon_k}(z) = f(z) + \varepsilon_k^2$ with

$$\varepsilon_k = \sqrt{\frac{\pi^2}{4(n^2+k)}} \simeq \frac{\pi}{2n+k/n} \quad \text{and} \quad 1 \leq k \leq 2n+1.$$

Quiz

What did Hamal say after eating a pie that he cooked himself ?

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Quiz

What did Hamal say after eating a pie that he cooked himself ?
That is the best pie I have ever eaten in my entire life.

The constant $\pi^2/4$

- $g(w) = w + c_2 w^2 + \mathrm{O}(w^3)$,
- $f(z) = z - a_2 z^2 + \mathrm{O}(z^3)$,
- $P(w, z) = (g(w), f(z) + b_1 w)$.

- The constant b_1 is not invariant under local conjugacy.

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- The invariant is

$$b := \frac{a_2 b_1}{c_2} =: \frac{\pi^2}{4\alpha^2}.$$

Lemma

Assume (T_n), (T'_n) are sequences of integers such that $\sigma_n := T'_n - (\sqrt{T_n} + \alpha)^2$ is bounded. Then as $n \to \infty$,

$$P^{\circ T'_n - T_n}(g^{\circ T_n}(w), z) = \left(g^{\circ T'_n}(w), \mathcal{L}_{\sigma_n}(z) + \mathrm{o}(1)\right).$$

The choice of α ?

Question

How can we choose α, $T_n' = T_{n+1}$ and control

$$\sigma_n := T_{n+1} - \left(\sqrt{T_n} + \alpha\right)^2 \quad ?$$

- In our proof, we chose

$$\alpha := 1 \quad \text{and} \quad T_n := n^2 \quad \text{so that} \quad b = \frac{\pi^2}{4} \quad \text{and} \quad \sigma_n = 0.$$

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Where did Hamal buy the deer that he cooked for christmas dinner ?

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Where did Hamal buy the deer that he cooked for christmas dinner ?
He found it dead on the road during his bike ride.

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$$\sigma_n := T_{n+1} - \left(\sqrt{T_n} + \alpha\right)^2 \quad ?$$

- With Matthieu Astorg we realized that we could take

$$\alpha := k \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad T_n := k^2 n^2$$

so that

$$b = \frac{\pi^2}{4k^2} \quad \text{and} \quad \sigma_n = 0.$$

The choice of α ?

Question

How can we choose α, $T'_n = T_{n+1}$ and control

$$\sigma_n := T_{n+1} - \left(\sqrt{T_n} + \alpha\right)^2 \quad ?$$

- Yesterday, with Matthieu Astorg and Yusheng Luo, we realized that we could have chosen

$$\alpha := \sqrt{\frac{k}{2}} \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad T_n := \frac{kn(n-1)}{2}$$

so that

$$b = \frac{\pi^2}{2k} \quad \text{and} \qquad \sigma_n = kn\left(1 - \frac{1}{2n} - \sqrt{1 - \frac{1}{n}}\right) \to 0.$$

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Did Hamal tell you what Hamal stands for ?

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Did Hamal tell you what Hamal stands for ?
It is among the brightest stars in the night sky, but only the **second** brightest in the constellation Aries.

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Happy birthday, Hamal ! !