

1 Product Metrics

Definition 1.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Then their Cartesian product $X \times Y$ has a metric space structure, defined by the metric

$$d_{X \times Y} : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

We will call $d_{X \times Y}$ the *product metric* on $X \times Y$.¹

Example 1.2. Consider \mathbb{R} with the Euclidean metric. What is the product metric on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$?

In-class Exercises

- Verify that $d_{X \times Y}$ does in fact define a metric on $X \times Y$.
- (a) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.
 (b) Let $U \subseteq X \times Y$ be an open set, and let $(x, y) \in U$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq U$.
- Definition (Projection maps).** For a product of sets $X \times Y$, the maps

$$\begin{array}{ll} \pi_X : X \times Y \rightarrow X & \pi_Y : X \times Y \rightarrow Y \\ \pi_X(x, y) = x & \pi_Y(x, y) = y \end{array}$$

are called the *projection onto X* and the *projection onto Y* , respectively.

- (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and endow their product $X \times Y$ with the product metric $d_{X \times Y}$. Show that the projection map

$$\pi_X : (X \times Y, d_{X \times Y}) \longrightarrow (X, d_X)$$

is continuous. (The same argument, which you do not need to repeat, shows that the map π_Y is continuous).

- (b) **Definition (Open maps).** A map $f : W \rightarrow Z$ of metric spaces is called *open* if $f(U) \subseteq Z$ is open for every open subset $U \subseteq W$. In other words, the image of every open subset is open.

Prove that the projection map π_X is open. (The same argument shows π_Y is open).

Hint: Use Question 2.

¹*Remark:* In light of Problem 6(b), any of the metrics in Problem 6 – among others – are sometimes called the *product metrics* on $X \times Y$. For clarity in this course we will use this term specifically for the metric $d_{X \times Y}$.

4. **(Optional)**. Compare and contrast the definition of an **open** map with the definition of a **continuous** map. Find examples of maps $f : X \rightarrow Y$ that are ...

- | | |
|-----------------------------|---------------------------------|
| (a) open but not continuous | (c) open and continuous |
| (b) continuous but not open | (d) neither continuous nor open |

5. **(Optional)**. A map is called *closed* if the image of every closed set is closed. Prove or find a counterexample: the projection map $\pi_X : X \times Y \rightarrow X$ is always a closed map.

6. **(Optional)**. Let (X, d_X) and (Y, d_Y) be metric spaces.

(a) Show that each of the following functions also defines a metric on $X \times Y$.

$$d_1 : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$

$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

$$d_\infty : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

(b) Show that these metrics on $X \times Y$ are all *topologically equivalent* to the metric $d_{X \times Y}$. This means that a subset of $X \times Y$ open with respect to one metric if and only if it is open with respect to the other.

7. **(Optional)**. Let (X_i, d_i) be metric spaces for $i \in \mathbb{N}$. Can you find a natural way to define a metric on the infinite product $\prod_{i \in \mathbb{N}} X_i = X_1 \times X_2 \times X_3 \times \dots$?

There are multiple ways to do this, which are not topologically equivalent!

8. **(Optional)**. Let (X, d) be a metric space. Show that the map $d : X \times X \rightarrow \mathbb{R}$ is continuous with respect to the product metric on $X \times X$ and the standard topology on \mathbb{R} .