## 1 The interior and the closure of a set

**Definition 1.1. (Interior of a set.)** Let $(X, d)$ be a metric space, and $A \subseteq X$ a subset. Then the *interior of* $A$, denoted $\text{Int}(A)$ or $\mathring{A}$ , is defined to be the set

$$\text{Int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}.$$

Note that $\text{Int}(A) \subseteq A$. We will see in the exercises that $\text{Int}(A)$ is an open set, and it is in a sense the largest open subset of $A$.

**Definition 1.2. (Closure of a set.)** Let $(X, d)$ be a metric space, and $A \subseteq X$ a subset. Then the *closure of* $A$, denoted $\overline{A}$, is defined to be the set

$$\overline{A} = \{x \in X \mid \text{ for every } r > 0 \text{ the ball } B_r(x) \text{ contains a point of } A\}.$$

We will see that $\overline{A}$ is a closed set, and that in a sense it is the smallest closed set containing $A$.

**Example 1.3.** What is the closure of the open set $B_1(0, 0) \subseteq \mathbb{R}^2$?

## In-class Exercises

1. For this problem we introduce the following terminology.

   **Definition 1.4. (Neighbourhood of a point $x$.)** Let $(X, d)$ be a metric space, and $x \in X$. Then any open set $U$ containing $x$ is called an *open neighbourhood of $x$*, or simply a *neighbourhood of $x$*.

   (a) Prove the following.
   **Theorem 1.5. (Equivalent definition of interior point.)** *For a subset $V$ of a metric space $X$, a point $x \in V$ is an interior point of $V$ if and only if there exists an open neighbourhood $U$ of $x$ that is contained in $V$.*

   (b) Prove the following.
   **Theorem 1.6. (Equivalent definition of closure.)** *For a subset $A$ of a metric space $X$, the closure of $A$ is equal to the set*

   $$\overline{A} = \{x \in X \mid \textit{every neighbourhood } U \textit{ of } x \textit{ contains a point of } A\}.$$

2. Prove the following theorem.

**Theorem 1.7.** *Let $(X, d)$ be a metric space, and $A \subseteq X$ a subset.*

*(i)* $\operatorname{Int}(A) \subseteq A$

*(ii)* *$A$ is open if and only if* $A = \operatorname{Int}(A)$

*(iii)* *If $A \subseteq B$ then* $\operatorname{Int}(A) \subseteq \operatorname{Int}(B)$

*(iv)* $\operatorname{Int}(\operatorname{Int}(A)) = \operatorname{Int}(A)$

*(v)* $\operatorname{Int}(A)$ *is open in $X$*

*(vi)* $\operatorname{Int}(A)$ *is the largest open subset of $A$ in the following sense: If $U \subseteq A$ is any open subset of $A$, then* $U \subseteq \operatorname{Int}(A)$

3. Prove the following theorem.

**Theorem 1.8.** *Let $(X, d)$ be a metric space, and $A \subseteq X$ a subset.*

*(i)* $A \subseteq \overline{A}$

*(ii)* *If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$*

*(iii)* *$A$ is closed if and only if $A = \overline{A}$*

*(iv)* $\overline{\overline{A}} = \overline{A}$

*(v)* *$\overline{A}$ is closed in $X$*

*(vi)* *$\overline{A}$ is the smallest closed set containing $A$, in the following sense: If $A \subseteq C$ for some closed set $C$, then $\overline{A} \subseteq C$*

4. **(Optional).** Let $A$ be a subset of a metric space $(X, d)$. Explore the relationships between the sets

$$\operatorname{Int}(X \setminus A) \qquad X \setminus \operatorname{Int}(A) \qquad \overline{X \setminus A} \qquad X \setminus \overline{A}$$

Determine which of these sets are necessarily equal or necessarily subsets of one another. Give counterexamples to show where equality or containment fails.

5. **(Optional).** Let $A_i$, $i \in I$, be a collection of subsets of a metric space $(X, d)$. For each of the following statements, either prove the statement, or construct a counterexample.

(a) $\operatorname{Int}\left(\bigcup_{i \in I} A_i\right) \subseteq \bigcup_{i \in I} \operatorname{Int}(A_i)$

(b) $\operatorname{Int}\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} \operatorname{Int} A_i$

(c) $\operatorname{Int}\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} \operatorname{Int}(A_i)$

(d) $\operatorname{Int}\left(\bigcap_{i \in I} A_i\right) \supseteq \bigcap_{i \in I} \operatorname{Int}(A_i)$

(e) $\overline{\bigcup_{i \in I} A_i} \subseteq \bigcup_{i \in I} \overline{A_i}$

(f) $\overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \overline{A_i}$

(g) $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}$

(h) $\overline{\bigcap_{i \in I} A_i} \supseteq \bigcap_{i \in I} \overline{A_i}$

6. **(Optional).** Prove the following equivalent definition of continuity.

**Theorem (An equivalent definition of continuity).** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Then a map $f : X \to Y$ is continuous if and only if

$$f\left(\overline{A}\right) \subseteq \overline{f(A)} \qquad \text{for every subset } A \subseteq X.$$

7. **(Optional).** For a metric $(X, d)$, let $x_0 \in X$ and $r > 0$. You proved on the homework that the set

$$C_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$$

is closed. Explain why $C_r(x_0)$ always contains the closure of the ball $B_r(x_0)$. Give an example of a metric space where $C_r(x_0)$ is equal to $\overline{B_r(x_0)}$ for every $r > 0$ and $x_0$, and give an example of a metric space and $x_0, r$ such that $C_r(x_0)$ is a strict subset of $\overline{B_r(x_0)}$.