

## 1 Convergent sequences in metric spaces

**Definition 1.1. (Convergent sequences in  $\mathbb{R}$ .)** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers. Then we say that the sequence *converges* to  $a_\infty \in \mathbb{R}$ , and write  $\lim_{n \rightarrow \infty} a_n = a_\infty$ , if ...

**Definition 1.2. (Convergent sequences in metric spaces.)** Let  $(X, d_X)$  be a metric space, and let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $X$ . Then we say that the sequence *converges* to  $a_\infty \in X$ , and write  $\lim_{n \rightarrow \infty} a_n = a_\infty$ , if ...

Rephrased:

### In-class Exercises

1. Prove the following result:

**Theorem (An equivalent definition of convergence.)** A sequence  $(a_n)_{n \in \mathbb{N}}$  of points in a metric space  $(X, d)$  converges to  $a_\infty$  if and only if for any open set  $U \subseteq X$  which contains  $a_\infty$ , there exists some  $N > 0$  so that  $a_n \in U$  for all  $n \geq N$ .

2. (Uniqueness of limits in a metric space).

(a) Let  $(X, d)$  be a metric space, and let  $x, y$  be distinct points in  $X$ . Show that there exist **disjoint** open subsets  $U_x$  and  $U_y$  of  $X$  such that  $x \in U_x$  and  $y \in U_y$ .

*Remark:* This is called the *Hausdorff* property of metric spaces, and this result shows that metric spaces are  $T_2$ -spaces.

(b) Let  $(X, d)$  be a metric space. Show that the limit of a sequence, if it exists, is **unique**, in the following sense. Suppose that  $(a_n)_{n \in \mathbb{N}}$  is a sequence in  $X$  that converges to a point  $a_\infty \in X$ , and converges to a point  $\tilde{a}_\infty \in X$ . Show that  $a_\infty = \tilde{a}_\infty$ .

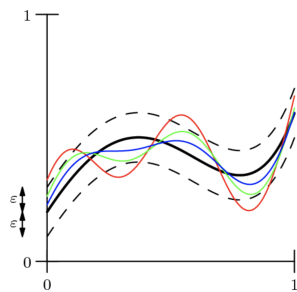
3. **(Optional)** Let  $(X, d)$  be a metric space. Let  $(a_n)$  be a sequence in  $X$  that converges to a point  $a_\infty \in X$ . Show that the set  $\{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$  is a closed subset of  $X$ .

4. **(Optional) Definition (Pointwise and Uniform Convergence).** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of functions  $f_n : X \rightarrow Y$ .

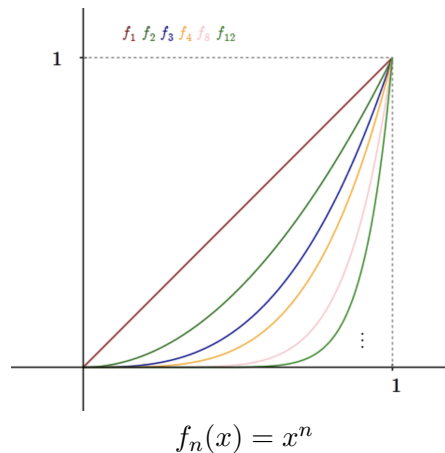
- The sequence  $(f_n)_{n \in \mathbb{N}}$  *converges at a point*  $x \in X$  if the sequence  $(f_n(x))_{n \in \mathbb{N}}$  of points in  $Y$  converges.
- The sequence  $(f_n)_{n \in \mathbb{N}}$  *converges pointwise* to a function  $f_\infty : X \rightarrow Y$  if for every point  $x \in X$  the sequence  $(f_n(x))_{n \in \mathbb{N}}$  of points in  $Y$  converges to the point  $f_\infty(x) \in Y$ .
- The sequence  $(f_n)_{n \in \mathbb{N}}$  *converges uniformly* to a function  $f_\infty : X \rightarrow Y$  if for every  $\epsilon > 0$  there is some  $N \in \mathbb{N}$  so that  $d_Y(f_n(x), f_\infty(x)) < \epsilon$  for every  $n \geq N$  and  $x \in X$ .

In other words, if the sequence  $(f_n)_{n \in \mathbb{N}}$  converges pointwise to  $f_\infty$ , then for each  $\epsilon > 0$  the choice of  $N$  may depend on the point  $x \in X$ . To converge uniformly to  $f_\infty$ , there must exist a choice of  $N$  that is independent of the point  $x$ .

- (a) Use the following picture of functions  $f_1$ ,  $f_2$ ,  $f_3$ , and  $f_\infty$  to explain the concept of uniform convergence of functions  $\mathbb{R} \rightarrow \mathbb{R}$ , and how it differs from pointwise convergence.



- (c) Consider the space  $[0, 1]$  with the usual metric, and the sequence of functions  $f_n : [0, 1] \rightarrow [0, 1]$  defined by  $f_n(x) = x^n$ . Show that this sequence converges pointwise, but conclude from part (b) that it does not converge uniformly.



- (b) Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of continuous functions  $f_n : X \rightarrow Y$  that converges uniformly to a function  $f_\infty : X \rightarrow Y$ . Show that  $f_\infty$  is continuous.

- (d) Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of functions  $f_n : X \rightarrow Y$ . Show that uniform convergence implies pointwise convergence.

- (e) Recall the metric on the space  $\mathcal{C}(a, b)$  of continuous functions  $[a, b] \rightarrow \mathbb{R}$ ,

$$d_\infty : \mathcal{C}(a, b) \times \mathcal{C}(a, b) \longrightarrow \mathbb{R}$$

$$d_\infty(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|.$$

Consider a sequence of continuous functions  $f_n \in \mathcal{C}(a, b)$ . Show that  $(f_n)_{n \in \mathbb{N}}$  converges with respect to the metric  $d_\infty$  if and only if it converges uniformly.

- (f) **(Challenge)** Is there a metric on  $\mathcal{C}(a, b)$  where convergence of a sequence is equivalent to pointwise convergence?