1 Continuous functions on metric spaces

Definition 1.1. (Continuous functions $f : \mathbb{R} \to \mathbb{R}$.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is continuous at a point $x \in \mathbb{R}$ if . . .

The function f is called *continuous* if it is continuous at every point $x \in \mathbb{R}$.

Rephrased:

How can we generalize this definition to general metric spaces?

Definition 1.2. (Continuous functions on metric spaces.) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \to Y$ be a function. Then f is continuous at a point $x \in X$ if ...

The function f is called *continuous* if it is continuous at every point $x \in X$.

Rephrased:

Notation: Let $f: X \to Y$ be a function.

For a subset $A \subseteq X$, the *image* of A is the set $f(A) = \{f(a) \mid a \in A\}$, a subset of Y.

For a subset $B \subseteq Y$, the preimage of B is the set $f^{-1}(B) = \{x \mid f(x) \in B\}$, a subset of X.

This definition makes sense (and we use the notation $f^{-1}(B)$) even when f is not invertible.

In-class Exercises

1. In this question, we will prove the following result:

Theorem (Continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: X \to Y$ be a function. Then f is continuous if and only if, given any open set $U \subseteq Y$, its preimage $f^{-1}(U) \subseteq X$ is open.

- (a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f: X \to Y$ be a continuous function. Suppose that $U \subseteq Y$ is an open set. Prove that $f^{-1}(U)$ is open.
- (b) Suppose that f is a function with the property that, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is an open set in X. Show that f is continuous.
- 2. Let $f: X \to Y$ and $g: Y \to Z$ be continuous functions between metric spaces. Show that the composite

$$g \circ f : X \to Z$$

is continuous. Hint: With our new criterion for continuity, this argument can be quite quick!

3. (Optional) Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is called an isometric embedding if it is "distance-preserving" in the sense that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$
 for all $x_1, x_2 \in X$.

- (a) Give an intuitive description, with pictures, of what it means for a map to be an isometric embedding.
- (b) Show that an isometric embedding is continuous.
- (c) Show that an isometric embedding is always injective.
- (d) Show that map

$$f: \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$x \longmapsto (x, 0)$$

is an isometric embedding of \mathbb{R} into \mathbb{R}^2 (each with the Euclidean metric).

- (e) Consider the function $f: \mathbb{R} \to \mathbb{R}^2$ given by the map f(x) = (x, mx + b) for $m, b \in \mathbb{R}$, so f maps the real line to the graph of the function mx + b. For which values of m and b is this an isometric embedding of Euclidean spaces?
- (f) For functions f(x) = (x, mx+b) that are not isometric embeddings, can you find a different parameterization of this line that is an isometric embedding? In other words, can you find an isometric embedding $g : \mathbb{R} \to \mathbb{R}^2$ whose image is the set $\{(x, mx+b) \mid x \in \mathbb{R}\}$?
- (g) Show that the image of any isometric embedding from \mathbb{R} into \mathbb{R}^2 must be a straight line.
- (h) Let $X = \{a, b, c\}$ be a 3-point set. Find examples of metrics on X so that the resulting metric space can and cannot be isometrically embedded in Euclidean space \mathbb{R}^2 . Can you find necessary and sufficient conditions on the metric on X to guarantee the existence of an isometric embedding of X into \mathbb{R}^2 ?