1 Connected and path-connected topological spaces

Definition 1.1. (Path-connected spaces). Let $(X, \mathcal{T})$ be a topological space, and let $x, y \in X$. Recall that a *path from x to y* is a continuous function $\gamma : ([0,1], \text{Euclidean}) \to (X, \mathcal{T})$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A space $(X, \mathcal{T})$ is *path-connected* if, given any two points $x, y \in X$, there exists a path from x to y.

Example 1.2. Sierpiński space is the space $\mathbb{S} = \{0, 1\}$ with the topology $\mathcal{T} = \{\varnothing, \{0\}, \{0, 1\}\}$. Is $\mathbb{S}$ path-connected?

In-class Exercises

1. In this problem, we will prove the following result:

   Theorem (Connectivity of product spaces). Let X and Y be nonempty topological spaces. Then the product space $X \times Y$ (with the product topology) is connected if and only if both X and Y are connected.

   Hint: See Worksheet #15, Problem 6(b).

   (a) Suppose that $X \times Y$ is nonempty and connected in the product topology $\mathcal{T}_{X\times Y}$. Prove that X and Y are connected.

   (b) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces, and suppose that $(a, b) \in X \times Y$. Prove that $\Big(X \times \{b\}\Big) \cup \Big(\{a\} \times Y\Big)$ is a connected subset of the product $X \times Y$ with the product topology $\mathcal{T}_{X\times Y}$.

   (c) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces. Prove that $X \times Y$ is connected in the product topology $\mathcal{T}_{X\times Y}$.

2. Let X be a (nonempty) topological space with the indiscrete topology. Is X necessarily path-connected?

3. Prove the following.

   Theorem (Path-connected $\implies$ connected). Let $(X, \mathcal{T})$ be a topological space. If X is path-connected, then X is connected.

   Hint: You may use the result from Homework #11 that the interval $[0, 1]$ is connected.

4. Let $f : X \to Y$ be a continuous map of topological spaces. Prove that if X is path-connected, then $f(X)$ is path-connected. In other words, the continuous image of a path-connected space is path-connected.

5. **(Optional).** Recall that $\mathcal{C}(0, 1)$ denotes the set of continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$, and that $\mathcal{C}(0, 1)$ is a metric space with metric

$$\begin{aligned} d_\infty : \mathcal{C}(0,1) \times \mathcal{C}(0,1) &\longrightarrow \mathbb{R} \\ d(f, g) &= \sup_{x\in[0,1]} |f(x) - g(x)| \end{aligned}$$

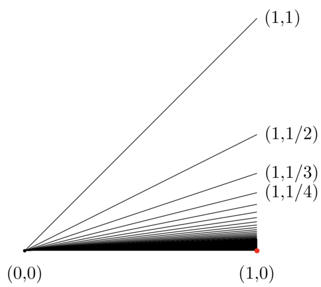
   Show that this metric space is path-connected, and therefore connected.

6. **(Optional).** This problem shows that the converse to Problem 3 fails.

Let X be the following subspace of $\mathbb{R}^2$ (with topology induced by the Euclidean metric)

$$X = \{(1,0)\} \cup \bigcup_{n\in\mathbb{N}} L_n,$$

where L_n is the closed line segment connecting the origin $(0,0)$ to the point $(1, \frac{1}{n})$.



(a) Show that X is connected.

(b) **(Challenge).** Show that X is **not** path-connected.

(c) Would the space be path–connected if we added in the line segment from $(0,0)$ to $(1,0)$?

7. **(Optional).**

Definition (Local connectedness). Let $(X, \mathcal{T})$ be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a connected open neighbourhood V_x of x . The space X is *locally connected* if it is locally connected at every point $x \in X$.

Definition (Local path-connectedness). Let $(X, \mathcal{T})$ be a topological space. Then X is *locally connected at a point* $x \in X$ if every neighbourhood U_x of x contains a path-connected open neighbourhood V_x of x . The space X is *locally path-connected* if it is locally path-connected at every point $x \in X$.

(a) Let $(X, \mathcal{T})$ be a topological space, and let $x \in X$. Show that if X is locally path-connected at x, then it is locally connected at x. Conclude that locally path-connected spaces are locally connected.

(b) Let $X = (0,1) \cup (2,3)$ with the Euclidean metric. Show that X is locally path-connected and locally connected, but is not path-connected or connected.

(c) Let X be the following subspace of $\mathbb{R}^2$ (with topology induced by the Euclidean metric)

$$X = \bigcup_{n\in\mathbb{N}} \left(\{\tfrac{1}{n}\} \times [0,1]\right) \quad \bigcup \quad \left(\{0\} \times [0,1]\right) \quad \bigcup \quad \left([0,1] \times \{0\}\right).$$

Show that X is path-connected and connected, but not locally connected or locally path-connected.

(d) **(Challenge).** Consider the natural numbers $\mathbb{N}$ with the cofinite topology. Show that $\mathbb{N}$ is locally connected but not locally path-connected.