

1 Connected topological spaces

Definition 1.1. (Disconnected spaces; connected spaces). A topological space (X, \mathcal{T}) is *disconnected* if there exist disjoint nonempty open subsets A and B in X such that $X = A \cup B$. We call the sets A and B a *separation* of X . If no separation of X exists, then X is called *connected*.

A subset S of X is said to be *connected* if it is connected when viewed with the subspace topology (S, \mathcal{T}_S) . This means ...

Example 1.2. Determine which of the following topological spaces are connected.

(a) $X = \mathbb{R}$, $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$

(b) $X = \{a, b, c, d\}$, $\mathcal{T} = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}, \{d\}, \{a, b, d\}, \{c, d\}, \{a, b, c, d\}\}$

In-class Exercises

1. Show that the following topological spaces (with the Euclidean metric) are disconnected.

(a) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(b) $(0, 1) \cup \{5\}$

(c) \mathbb{Q}

2. (a) Give an example of a connected topological space X , and a subset $S \subseteq X$ that is disconnected.

(b) Give an example of a disconnected topological space X , and a subset $S \subseteq X$ that is connected.

3. Prove that a topological space (X, \mathcal{T}) is disconnected if and only if there is subset A , with $\emptyset \subsetneq A \subsetneq X$, that is both open and closed.

4. Consider $\{0, 1\}$ as a topological space with the discrete topology. Show that a topological space (X, \mathcal{T}) is disconnected if and only if there is a continuous **surjective** function $X \rightarrow \{0, 1\}$.

5. Prove the following (often useful) lemma:

Lemma. Let X be a topological space, and let A, B be a separation of X . Let $S \subseteq X$. If S is connected, then $S \subseteq A$ or $S \subseteq B$.

6. Let X be a topological space, and let A_i , $i \in I$, be a collection of subsets of X . Suppose that A_i is connected for each i .

(a) Show by example that the union $\bigcup_{i \in I} A_i$ may be disconnected.

(b) Suppose that $\bigcap_{i \in I} A_i \neq \emptyset$. Show that the union $\bigcup_{i \in I} A_i$ is connected.

7. **(Optional)**. Consider \mathbb{R} with the Euclidean metric. Which of the following subsets are connected?

$$\{x \in \mathbb{R} \mid d(x, 1) < 1 \text{ or } d(x, -1) < 1\}$$

$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) < 1\}$$

$$\{x \in \mathbb{R} \mid d(x, 1) \leq 1 \text{ or } d(x, -1) \leq 1\}$$

8. **(Optional)**. Consider the set $X = \{a, b, c, d\}$. For which of the following topologies \mathcal{T} is the topological space (X, \mathcal{T}) connected?

(a) $\mathcal{T} = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$

(b) $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{a, b, c, d\}\}$

9. **(Optional)**. Consider the following topologies on \mathbb{R} . Which of these topological spaces are connected?

(a) indiscrete topology

(b) discrete topology

(c) Euclidean topology

(d) cofinite topology

(e) $\mathcal{T} = \{\mathbb{R}, (0, 1), \emptyset\}$

(f) $\mathcal{T} = \{\mathbb{R}, \{0, 1\}, \{0\}, \{1\}, \emptyset\}$

(g) $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$

(h) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \in A\} \cup \{\emptyset\}$

(i) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 0 \notin A\} \cup \{\mathbb{R}\}$

(j) $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, 1 \in A\} \cup \{\emptyset\}$

10. **(Optional)**. Let (X, \mathcal{T}) be a topological space. Let $\{A_n \mid n \in \mathbb{Z}\}$ be a family of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$ for every n . Prove $\bigcup_{n \in \mathbb{Z}} A_n$ is connected.
11. **(Optional)**. Let (X, \mathcal{T}) be a topological space. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of connected subspaces in X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$. Is $\bigcap_{n \in \mathbb{N}} A_n$ necessarily connected?
12. **(Optional)**. Let (X, \mathcal{T}) be a topological space, and let $A, B \subseteq X$. Suppose $A \cup B$ and $A \cap B$ are connected. Prove that if A and B are both closed or both open, then A and B are connected.