1 Products of topological spaces

Definition 1.1. (The product topology.) Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Then the product topology $\mathcal{T}_{X\times Y}$ on $X\times Y$ is the collection of subsets of $X\times Y$ generated by the set

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ is open, and } V \subseteq Y \text{ is open}\}.$$

This means that $\mathcal{T}_{X\times Y}$ consists of all unions of elements of \mathcal{B} .

Proposition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Then the product metric $d_{X \times Y}$ induces the product topology on $X \times Y$.

In-class Exercises

- 1. Verify that $\mathcal{T}_{X\times Y}$ is indeed a topology on $X\times Y$, and that \mathcal{B} is a basis for this topology. Hint: One approach is to use Worksheet #12 Problem 5.
- 2. Prove that the projection map $\pi_X : X \times Y \to X$ is both continuous and open with respect to the topologies $\mathcal{T}_{X \times Y}$ and \mathcal{T}_X . (The same argument shows that the projection map π_Y is both continuous and open).
- 3. Let $f_1: X \to Y_1$ and $f_2: X \to Y_2$ be two functions of topological spaces, and define a function

$$f: X \to Y_1 \times Y_2$$

$$f(x) = (f_1(x), f_2(x))$$

Show that f is continuous (with respect to the product topology on $Y_1 \times Y_2$) if and only if both f_1 and f_2 are continuous.

Hint: Notice $f_i = \pi_i \circ f$, where $\pi_1 : Y_1 \times Y_2 \to Y_1$, $\pi_2 : Y_1 \times Y_2 \to Y_2$ are the projection maps.

4. (Optional). Prove the following theorem.

Theorem (Equivalent definition of the product topology). Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Then the product topology $\mathcal{T}_{X\times Y}$ on $X\times Y$ is precisely the collection of subsets of $X\times Y$,

$$\mathcal{T}_{X\times Y} = \left\{ \begin{array}{c} W \mid & \text{for each } (x,y) \in W, \text{ there is a some } U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \\ & \text{such that } (x,y) \in (U \times V) \subseteq W \end{array} \right\}.$$

5. (Optional). (Finite products). Given a finite product $X = X_1 \times X_2 \times \cdots \times X_n$ of topological spaces, we can define a topology on X by induction, by first taking the product topology on $X_1 \times X_2$, then the product topology on $(X_1 \times X_2) \times X_3$, etc. Show that the resultant topology on X (called the *product topology*) is generated by the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \subseteq X_i \text{ is open}\}.$$

- 6. (Optional). (Infinite products). Let $\{X_i\}_{i\in I}$ be a (possibly infinite) collection of sets, and let $X = \prod_{i\in I} X_i$ be their product. We denote elements of X by $(x_i)_{i\in I}$. Define two topologies on X:
 - The box topology on X is the topology generated by the basis

$$\mathcal{B}_B = \left\{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ is open} \right\}.$$

• The product topology on X is the topology generated by the basis

$$\mathcal{B}_P = \left\{ \prod_{i \in I} U_i \;\middle|\; U_i \subseteq X_i \text{ is open, } U_i = X_i \text{ for all but finitely many } i \in I \right\}.$$

- (a) Prove that both \mathcal{B}_B and \mathcal{B}_P are bases (in the sense of Worksheet #12, Problem 5). Conclude that the box topology and product topology on X are, in fact, topologies.
- (b) Suppose that I is finite. Show that the box topology and the product topology are equal, and both are the usual product topology on X in the sense of Problem 5.
- (c) Which topology is finer, the box or the product topology? If we consider maps from a topological space into the product X, what can you say about the relationship between continuity of a map with respect to the box topology, and continuity with respect to the product topology? What about for maps out of X?
- (d) For reasons that are formalized using "category theory" and the concept of a "universal property", we want our products to satisfy the following statement:

Let $\{Y_i\}_{i\in I}$ be a collection of topological spaces, and $\prod_{i\in I}Y_i$ their product. Let X be any topological space, and let $f_i:X\to Y_i$ be a collection of functions. Then the function

$$f: X \to \prod_{i \in I} Y_i$$

 $f(x) = (f_i(x))_{i \in I}$

is continuous if and only if each function f_i is continuous.

Prove that this property always holds if we put the product topology on $\prod_{i \in I} Y_i$, but that this property may fail if we put the box topology on $\prod_{i \in I} Y_i$. This property is the reason that the product topology is generally considered the "correct" topology on $\prod_{i \in I} Y_i$. Hint: Consider the function $f : \mathbb{R} \to \prod_{n \in \mathbb{N}} \mathbb{R}$ given by $f(x) = (x, x, x, \ldots)$.

(e) Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable collection of metric spaces. Show that the product topology on $\prod_{n\in\mathbb{N}} X_n$ is metrizable, but the box topology is not.