

## 1 Bases for topological spaces

**Definition 1.1. (Basis of a topology.)** Let  $(X, \mathcal{T})$  be a topological space. We say that a collection  $\mathcal{B}$  of subsets of  $X$  is a *basis* for the topology  $\mathcal{T}$  if

- $\mathcal{B} \subseteq \mathcal{T}$ , that is, every basis element is open, and
- every element of  $\mathcal{T}$  can be expressed as a union of elements of  $\mathcal{B}$ .

We say that the basis  $\mathcal{B}$  *generates* the topology  $\mathcal{T}$ .

**Remark 1.2.** By convention, we say that the empty set  $\emptyset$  is the union of an empty collection of open sets. So a basis  $\mathcal{B}$  does not need to include  $\emptyset$ .

**Example 1.3.** Let  $X$  be a set.

(a) Find a basis for the discrete topology on  $X$ .

(b) Find a basis for the indiscrete topology on  $X$ .

### In-class Exercises

1. **(The basis criteria).** Let  $(X, \mathcal{T})$  be a topological space. Show that a collection  $\mathcal{B}$  of subsets of  $X$  is a basis for  $\mathcal{T}$  if and only if it satisfies the following two conditions:

- (i) Every basis element is an **open** set, that is,  $\mathcal{B} \subseteq \mathcal{T}$ .
- (ii) For every open set  $U \in \mathcal{T}$ , and every  $x \in U$ , there exists some element  $B \in \mathcal{B}$  such that  $x \in B$  and  $B \subseteq U$ .

2. Let  $(X, d)$  be a metric space. Show that the set of open balls

$$\mathcal{B} = \{ B_r(x_0) \mid x_0 \in X, r \in \mathbb{R}, r > 0 \}$$

is a basis for the topology induced by  $d$ .

3. Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Recall that  $X \times Y$  is then a metric space with metric

$$d_{X \times Y} : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

(a) Show that the set

$$\mathcal{B} = \{ U \times V \mid U \subseteq X \text{ is open, } V \subseteq Y \text{ is open} \}$$

forms a basis for the topology induced by  $d_{X \times Y}$ .

(b) Does every open set in  $X \times Y$  have the form  $U \times V$ , where  $U \subseteq X$  and  $V \subseteq Y$  are open?

4. An advantage of identifying a basis for a topology is that many topological statements can be reduced to statements about the basis. As an example, prove the following theorem.

**Theorem 1.4. (Equivalent definition of continuity).** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces, and let  $\mathcal{B}_Y$  be a basis for  $\mathcal{T}_Y$ . Prove that a map  $f : X \rightarrow Y$  is continuous if and only if for every open set  $U \in \mathcal{B}_Y$ , the preimage  $f^{-1}(U) \subseteq X$  is open.

5. In our definition of a basis, we began with a space with a given topology, and defined a basis to be a collection of open subsets satisfying certain properties. In many cases, however, we will wish to topologize our set  $X$  by first specifying a basis, and using the basis to define a topology on  $X$ . Prove the following theorem, which gives conditions on a collection of subsets  $\mathcal{B}$  of  $X$  that ensure it will generate a valid topology on  $X$ .

**Theorem 1.5. (An extrinsic definition of a basis).** Let  $X$  be a set and let  $\mathcal{B}$  be a collection of subsets of  $X$  such that

- $\bigcup_{B \in \mathcal{B}} B = X$
- If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$  then there is some  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq (B_1 \cap B_2)$ .

Let

$$\mathcal{T} = \{ U \mid U \text{ is a union of elements of } \mathcal{B} \}.$$

Then  $\mathcal{T}$  is a topology on  $X$ , and  $\mathcal{B}$  is a basis for  $\mathcal{T}$ . We say that  $\mathcal{T}$  is the topology generated by the basis  $\mathcal{B}$ .

6. (Optional).

- (a) Consider the topology on  $\mathbb{R}^n$  induced by the Euclidean metric. Prove that the following set is a basis for  $\mathbb{R}^n$ .

$$\mathcal{B} = \{ B_\epsilon(x) \mid \epsilon > 0, \epsilon \text{ is rational; } x \in \mathbb{R}^n, \text{ all coordinates } x_i \text{ of } x \text{ are rational.} \}$$

- (b) Show that  $\mathbb{R}^n$  has uncountably many open sets, but that the basis  $\mathcal{B}$  is countable.

7. (Optional). Let  $(X, d)$  be a metric space, and  $\mathcal{B}$  a basis for the topology  $\mathcal{T}_d$  induced by  $d$ .

- (a) Let  $S \subseteq X$  be a subset, and  $s \in S$ . Show that  $s$  is an interior point of  $S$  if and only if there is some element  $B \in \mathcal{B}$  such that  $s \in B$  and  $B \subseteq S$ .

- (b) Deduce that  $\text{Int}(S) = \bigcup_{B \in \mathcal{B}, B \subseteq S} B$ .

8. (Optional). **Definition (Subbases).** Let  $X$  be a set, and let  $\mathcal{S}$  be a collection of subsets of  $X$  whose union is equal to  $X$ . Then the *topology generated by the subbasis*  $\mathcal{S}$  is the collection of all arbitrary unions of all finite intersections of elements in  $\mathcal{S}$ . *Remark:* In contrast to a basis, we are permitted to take finite intersections of sets in a subbasis.

- (a) Show that the set  $\mathcal{T}$  generated by a subbasis  $\mathcal{S}$  really is a topology, and is moreover the coarsest topology containing  $\mathcal{S}$ .

- (b) Verify that  $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, a) \mid a \in \mathbb{R}\}$  is a subbasis for the standard topology on  $\mathbb{R}$ .

- (c) Prove the following proposition.

**Proposition.** Let  $f : X \rightarrow Y$  be a function of topological spaces, and let  $\mathcal{S}$  be a subbasis for  $Y$ . Then  $f$  is continuous if and only if  $f^{-1}(U)$  is open for every subbasis element  $U \in \mathcal{S}$ .