1 Continuous functions on topological spaces

Definition 1.1. (Continuous functions of topological spaces.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is called *continuous* if . . .

In-class Exercises

1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and fix $y_0 \in Y$. Show that the constant map

$$f : X \to Y$$
$$f(x) = y_0$$

   is continuous.

2. (a) Let X be a set, and let $\mathcal{T}_{\therefore}$ denote the discrete topology on X. Let $(Y, \mathcal{T}_Y)$ be a topological space. Show that any map $f : X \to Y$ of these topological spaces is continuous.

   (b) Let $(X, \mathcal{T}_X)$ be a topological space. Let Y be a set, and let $\mathcal{T}_\bullet$ denote the indiscrete topology on Y. Show that any map $f : X \to Y$ of these topological spaces is continuous.

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $S \subseteq X$. Let $f : X \to Y$ be a continuous function. Show that the restriction of f to S,

$$f|_S : S \to Y,$$

   is continuous with respect to the subspace topology on S.

4. Below are two results that you proved for metric spaces. Verify that each of these results holds for abstract topological spaces. This is a good opportunity to review their proofs!

   (a) **Theorem (Equivalent definition of continuity.)** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is continuous if and only if for every closed set $C \subseteq Y$, the set $f^{-1}(C)$ is closed.

   (b) **Theorem (Composition of continuous functions.)** Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Prove that the map $g \circ f : X \to Z$ is continuous.

5. **(Optional).**
   (a) Let $f : X \to Y$ be a map of two topological spaces, and suppose that Y has the cofinite topology. Prove that f is continuous if and only if the preimage of every point in Y is closed in X.
   (b) Let $f : X \to Y$ be a map of two topological spaces that both have the cofinite topology. Prove that f is continuous if and only if either f is a constant function or the preimage of every point in Y is finite.

6. **(Optional).** Consider the following functions $f : \mathbb{R} \to \mathbb{R}$.
   (a) $f(x) = x$
   (b) $f(x) = 0$
   (c) $f(x) = x^2$
   (d) $f(x) = \cos(x)$
   (e) $f(x) = x + 1$
   (f) $f(x) = -x$

   Determine whether these functions are continuous when both the domain and codomain $\mathbb{R}$ have the topology ...

   - standard (Euclidean) topology
   - $\mathcal{T} = \{\mathbb{R}, \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
   - cofinite topology
   - $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$

7. **(Optional).** Let $X = \{a, b, c\}$ be the topological space with the topology

$$\{\varnothing, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

   Let $\mathbb{R}$ be the topological space defined by the usual Euclidean metric. Which of the following functions $f : \mathbb{R} \to X$ are continuous?

   (i) $f(x) = b$ for all $x \in \mathbb{R}$.

   (ii) $f(x) = \begin{cases} a, & x = (-\infty, 0) \\ b, & x = 0 \\ c, & x \in (0, \infty) \end{cases}$

   (iii) $f(x) = \begin{cases} a, & x = 0 \\ b, & x \in (-\infty, 0) \\ c, & x \in (0, \infty) \end{cases}$

   (iv) $f(x) = \begin{cases} a, & x \in (-\infty, 0] \\ b, & x \in (0, \infty) \end{cases}$

8. **(Optional).** Let $f : X \to Y$ be a function of topological spaces. Show that f is continuous if and only if it is continuous when restricted to be a map $f : X \to f(X)$, with $f(X) \subseteq Y$ given the subspace topology.

9. **(Optional).** Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that the identity map

$$id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$
$$id_X(x) = x$$

   is continuous with respect to the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.