1 Subspaces of topological spaces

Definition 1.1. (Subspace topology.) Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be any subset. Then S inherits the structure of a topological space, defined by the topology

$$\mathcal{T}_S = \{U \cap S \mid U \in \mathcal{T}_X\}.$$

The topology $\mathcal{T}_S$ on S is called the *subspace topology*.

Example 1.2. Describe the subspace topology on the following subsets of $\mathbb{R}$, with the topology induced by the Euclidean metric (we call this the "standard topology").

(a) $S = \{0, 1, 2\}$

(b) $S = (0, 1)$

In-class Exercises

1. Verify that the subspace topology is, in fact, a topology.

2. Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be any subset. Let ι_S be the *inclusion map*

$$\iota_S : S \to X$$
$$\iota_S(s) = s$$

   Verify that the subspace topology on S is precisely the set $\{i_S^{-1}(U) \mid U \subseteq X \text{ is open}\}$.

   Remark: We haven't defined these terms, but we can summarize this result by the slogan "the subspace topology on S is the coarsest topology that makes the inclusion maps ι_S continuous".

3. Let $(X, \mathcal{T}_X)$ be a topological space, and let $S \subseteq X$ be a subset. Let $\mathcal{T}_S$ denote the subspace topology on S.
   (a) Show by example that an open subset of S (in the subspace topology $\mathcal{T}_S$) may not be open as a subset of X. In other words, show there could be a subset $U \subseteq S$ with $U \in \mathcal{T}_S$, $U \notin \mathcal{T}_X$.
   (b) Conversely, suppose that $U \subseteq S$ and U is open in X. Show that U **is** open in the subspace topology on S. In other words, for $U \subseteq S$, if $U \in \mathcal{T}_X$ then $U \in \mathcal{T}_S$.
   (c) Suppose that S is a an open subset of X. Show that a subset $U \subseteq S$ is open in S (with the subspace topology) if and only if it is open in X. In other words, whenever S is open and $U \subseteq S$, $U \in \mathcal{T}_S$ if and only if $U \in \mathcal{T}_X$.

4. Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be a subset endowed with the subspace topology $\mathcal{T}_S$. Show that a set $C \subseteq S$ is closed in S if and only if there is some set $D \subseteq X$ that is closed in X with $C = D \cap S$.

5. **(Optional).** Let $(X, \mathcal{T}_X)$ be a topological space, and let $Z \subseteq Y \subseteq X$ be subsets. Show that the subspace topology on Z as a subspace of X coincides with the subspace topology on X as a subspace of Y (with the subspace topology as a subset of X). Conclude that there is no ambiguity in how to topologize the subset Z – to refer to its "subspace topology" we do not need to specify whether Y or X is the ambient space.

6. **(Optional).** Let (X, d) be a metric space, and let $\mathcal{T}_d^X$ be the topology induced by the metric. Let $S \subseteq X$ be a subset. We now have two methods of constructing a topology on S: we can restrict the metric from X to S, and take the topology $\mathcal{T}_d^S$ induced by the metric. We can also take the subspace topology $\mathcal{T}_S$ defined by $\mathcal{T}_d^X$. Show that these two topologies on S are equal, so there is no ambiguity in how to topologize a subset of a metric space.

7. **(Optional).** Let (X, d) be a metric space with the metric topology $\mathcal{T}_d$. Show that the subspace topology on any **finite** subset of X is the discrete topology.

8. **(Optional).** Let $(X, \mathcal{T})$ be a topological space, and $S \subseteq X$ a subset endowed with the subspace topology.
   (a) Suppose X has the discrete topology. Must S have the discrete topology?
   (b) Suppose X has the indiscrete topology. Must S have the indiscrete topology?
   (c) Suppose X is metrizable. Is S metrizable?
   (d) Recall that a topological space is *Hausdorff* if every pair of points have disjoint open neighbourhoods. If X is Hausdorff, then must S be Hausdorff?
   (e) A space has the T_1 *property* if every singleton subset $\{x\}$ is closed. If X is T_1, then must S be T_1?
   (f) For which of the above does the converse hold?

   Remark: A property is called *hereditary* if, whenever a topological space has the property, all of its subspaces necessarily have the property.

9. **(Optional).** Consider $\mathbb{R}$ with the standard topology (that is, the topology induced by the Euclidean metric). For each of a the following statements, construct a nonempty subset S of $\mathbb{R}$ with that satisfies the description, or prove that none exists.
   (a) S is an infinite, closed subset of $\mathbb{R}$, and the subspace topology on S is discrete.
   (b) S is not a closed subset of $\mathbb{R}$, and the subspace topology on S is discrete.
   (c) S has the indiscrete topology.
   (d) The subspace topology on S consists of exactly 2 open subsets.
   (e) The subspace topology on S consists of exactly 3 open subsets.