TOTALLY REAL POINTS IN THE MANDELBROT SET

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Abstract. Recently, Noytaptim and Petsche proved that the only totally real parameters $c \in \mathbb{Q}$ for which $f_c(z) := z^2 + c$ is postcritically finite are 0, −1 and −2 [NP]. In this note, we show that the only totally real parameters $c \in \mathbb{Q}$ for which $f_c$ has a parabolic cycle are $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$ and $\frac{7}{4}$.

Introduction

Consider the family of quadratic polynomials $f_c : \mathbb{C} \to \mathbb{C}$ defined by

$$f_c(z) := z^2 + c, \quad c \in \mathbb{C}.$$ 

The Mandelbrot set $M$ is the set of parameters $c \in \mathbb{C}$ for which the orbit of the critical point 0 under iteration of $f_c$ remains bounded:

$$M := \{ c \in \mathbb{C} \mid \forall n \geq 1, f_c^n(0) \in \mathbb{D}(0, 2) \}.$$ 

Definition 1. A parameter $c \in \mathbb{C}$ is postcritically finite if the orbit of 0 under iteration of $f_c$ is finite.

Definition 2. A parameter $c \in \mathbb{C}$ is parabolic if $f_c$ has a periodic cycle with multiplier a root of unity.

Postcritically finite parameters and parabolic parameters are algebraic numbers contained in $M$. More precisely, $c \in \mathbb{C}$ is a postcritically finite parameter if and only if $c$ is an algebraic integer whose Galois conjugates all belong to $M$ (see [M] and [Bu]). In addition, if $c \in \mathbb{C}$ is a parabolic parameter, then $4c$ is an algebraic integer (see [Bo]); moreover, $4c$ is an algebraic unit in $\mathbb{Z}/2\mathbb{Z}$ (see [M, Remark 3.2]).

Definition 3. An algebraic number $c \in \mathbb{Q}$ is totally real if its Galois conjugates are all in $\mathbb{Q} \cap \mathbb{R}$.

Recently, Noytaptim and Petsche [NP] completely determined the totally real postcritically finite parameters.

Proposition 1 (Noytaptim-Petsche). The only totally real parameters $c \in \mathbb{Q}$ for which $z \mapsto z^2 + c$ is postcritically finite are $-2$, $-1$ and $0$.

Their proof relies on the fact that the Galois conjugates of a postcritically finite parameter are also postcritically finite parameters, thus contained in $M$, and on the fact that $M \cap \mathbb{R} = [-2, \frac{1}{2}]$ has small arithmetic capacity. In this note, we revisit their proof. We then determine the totally real parabolic parameters.

Proposition 2. The only totally real parameters $c \in \mathbb{Q}$ for which $z \mapsto z^2 + c$ has a parabolic cycle are $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$ and $\frac{7}{4}$.

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1. Postcritically finite parameters

We first revisit the proof of Noytaptim and Petsche [NP].

Proof of Proposition 1. Assume that $c$ is a totally real postcritically finite parameter. Then, $c$ and all of its Galois conjugates are real postcritically finite parameters, thus lie in the interval $[-2, 0]$. Indeed,

- for $c \in (0, \frac{1}{4})$, the orbit of 0 under iteration $f_c$ is infinite, converging to an attracting fixed point of $f_c$;
- for $c = \frac{1}{4}$, the orbit of 0 under iteration $f_c$ is infinite, converging to a parabolic fixed point of $f_c$ at $z = \frac{1}{2}$;
- for $c \in (-\infty, -2) \cup \left(\frac{1}{4}, +\infty\right)$ the orbit of 0 under iteration $f_c$ is infinite, converging to $\infty$.

Let $a$ be a solution of $a + 1/a = c$; that is, $a^2 - ca + 1 = 0$. Then $a$ is an algebraic integer of modulus 1 with nonpositive real part, and all of its Galois conjugates also have modulus 1 and nonpositive real part.

By Kronecker’s theorem, $a$ is a root of unity. And since the Galois conjugates of $a$ all have nonpositive real part, the only possibilities are the following:

- $a = -1$, which is mapped to $c = -2$;
- $a = e^{\pm 2\pi i/3}$, which is mapped to $c = -1$;
- $a = \pm i$, which is mapped to $c = 0$.

Therefore, the only postcritically finite parameters that are totally real are $-2, -1, 0$.

2. Parabolic parameters

We now present the proof of Proposition 2. Note that $c = \frac{1}{4}, c = -\frac{3}{4}, c = -\frac{5}{4}$ and $c = -\frac{7}{4}$ are indeed parabolic parameters. Indeed,

- $f_{\frac{1}{4}}$ has a fixed point with multiplier 1 at $z = \frac{1}{2}$;
- $f_{-\frac{3}{4}}$ has a fixed point with multiplier $-1$ at $z = -\frac{1}{2}$;
- $f_{-\frac{5}{4}}$ has a cycle of period 2 with multiplier $-1$ consisting of the two roots of $4z^2 + 4z - 1$;
- $f_{-\frac{7}{4}}$ has a cycle of period 3 with multiplier 1 consisting of the three roots of $8z^3 + 4z^2 - 18z - 1$.

Proof of Proposition 2. Assume that $c$ is a totally real parabolic parameter. Then, the Galois conjugates of $c$ also are parabolic parameters. Either $c = \frac{1}{4}, c = -\frac{3}{4}, c = -\frac{5}{4}$, or $c$ and all of its Galois conjugates lie in the interval $[-2, -\frac{5}{4})$. Indeed, a parabolic cycle must attract the orbit of 0 under iteration of $f_c$. However,

- for $c \in (-\frac{5}{4}, \frac{1}{4})$, the orbit of 0 under iteration $f_c$ converges to an attracting fixed point of $f_c$;
- for $c \in (-\frac{5}{4}, -\frac{3}{4})$ the orbit of 0 under iteration $f_c$ converges to an attracting cycle of period 2 of $f_c$;
- for $c \in (-\infty, -2) \cup \left(\frac{1}{4}, +\infty\right)$ the orbit of 0 under iteration $f_c$ converges to $\infty$.

Let us assume that $c \in [-2, -\frac{5}{4})$. Then, $b := 4c + 6$ and all of its Galois conjugates lie in the interval $[-2, 1)$. Let $a$ be a solution of $a + 1/a = b$; that is, $a^2 - ba + 1 = 0$. 

Then \(a\) is an algebraic integer of modulus 1 with real part less than \(\frac{1}{2}\), and all of its Galois conjugates also have modulus 1 and real part less than \(\frac{1}{2}\).

By Kronecker’s theorem, \(a\) is a root of unity. And since the Galois conjugates of \(a\) all have real part less than \(\frac{1}{2}\), the only possibilities are the following:

- \(a = -1, b = -2\) and \(c = -2\); this is not a parabolic parameter;
- \(a = e^{\pm 12\pi i/3}, b = -1\) and \(c = -\frac{7}{4}\); this is indeed a parabolic parameter;
- \(a = \pm i, b = 0\) and \(c = -\frac{3}{2}\); in that case \(4c = -6\) is not an algebraic unit in \(\mathbb{Z}/2\mathbb{Z}\) and so, \(c\) is not a parabolic parameter;
- \(a = e^{\pm 12\pi i/5}, b = 2\cos(\frac{2\pi}{5})\) and \(c = \frac{\sqrt{5}-13}{8}\); in this case, \(f_c\) has an attracting cycle of period 4 and so, \(c\) is not a parabolic parameter;
- \(a = e^{\pm 4\pi i/5}, b = 2\cos(\frac{4\pi}{5})\) and \(c = \mp \frac{\sqrt{5}-13}{8}\); then the Galois conjugate \(\frac{\sqrt{5}-13}{8}\) is not a parabolic parameter and so, \(c\) is not a parabolic parameter.

This completes the proof of the proposition.

Remark: the following proof that \(-\frac{3}{2}\) is not a parabolic parameter was explained to us by Valentin Huguin. It follows from [Bo] that for all \(n \geq 1\),

\[
\text{discriminant}(f_c^n(z) - z, z) = P_n(4c) \quad \text{with} \quad P_n(b) \in \mathbb{Z}[b] \quad \text{and} \quad \pm P_n \text{ monic.}
\]

As an example,

\[
P_1(b) = -b + 1, \quad P_2(b) = (b - 1)(b + 3)^3, \quad P_3(z) = (b - 1)(b + 7)^3(b^2 + b + 7)^4,
\]

and

\[
P_4(z) = (b - 1)(b + 3)^3(b + 5)^6(b^3 + 9b^2 + 27b + 135)(b^2 - 2b + 5)^5.
\]

Note that this yields an alternate proof that \(c = \frac{1}{4}, c = -\frac{3}{4}, c = -\frac{5}{4}\) and \(c = -\frac{7}{4}\) are parabolic parameters. In addition,

\[
P_n(0) = \text{discriminant}(z^n - z, z) \equiv 1 \pmod{2}.
\]

As a consequence

\[
P_n(-6) \equiv 1 \pmod{2}.
\]

Thus, for all \(n \geq 1\), the roots of \(f_{-\frac{3}{2}}^n(z) - z\) are simple, which shows that \(f_{-\frac{3}{2}}\) has no parabolic cycle.

**References**


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