TOTALLY REAL POINTS IN THE MANDELBROT SET

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Abstract. Recently, Noytaptim and Petsche proved that the only totally real parameters $c \in \overline{\mathbb{Q}}$ for which $f_c(z) := z^2 + c$ is postcritically finite are 0, -1 and -2 [NP]. In this note, we show that the only totally real parameters $c \in \overline{\mathbb{Q}}$ for which f_c has a parabolic cycle are $\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{5}{4}$ and $-\frac{7}{4}$.

Introduction

Consider the family of quadratic polynomials $f_c : \mathbb{C} \to \mathbb{C}$ defined by

$$f_c(z) := z^2 + c, \quad c \in \mathbb{C}.$$

The Mandelbrot set M is the set of parameters $c \in \mathbb{C}$ for which the orbit of the critical point 0 under iteration of f_c remains bounded:

$$M := \left\{ c \in \mathbb{C} \mid \forall n \geq 1,\ f_c^{\circ n}(0) \in \overline{D}(0,2) \right\}.$$

Definition 1. A parameter $c \in \mathbb{C}$ is *postcritically finite* if the orbit of 0 under iteration of f_c is finite.

Definition 2. A parameter $c \in \mathbb{C}$ is *parabolic* if f_c has a periodic cycle with multiplier a root of unity.

Postcritically finite parameters and parabolic parameters are algebraic numbers contained in M. More precisely, $c \in \mathbb{C}$ is a postcritically finite parameter if and only if c is an algebraic integer whose Galois conjugates all belong to M (see [M] and [Bu]). In addition, if $c \in \mathbb{C}$ is a parabolic parameter, then $4c$ is an algebraic integer (see [Bo]); moreover, $4c$ is an algebraic unit in $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$ (see [M, Remark 3.2]).

Definition 3. An algebraic number $c \in \overline{\mathbb{Q}}$ is *totally real* if its Galois conjugates are all in $\overline{\mathbb{Q}} \cap \mathbb{R}$.

Recently, Noytaptim and Petsche [NP] completely determined the totally real postcritically finite parameters.

Proposition 1 (Noytaptim-Petsche)**.** *The only totally real parameters $c \in \overline{\mathbb{Q}}$ for which $z \mapsto z^2 + c$ is postcritically finite are -2, -1 and 0.*

Their proof relies on the fact that the Galois conjugates of a postcritically finite parameter are also postcritically finite parameters, thus contained in M, and on the fact that $M \cap \mathbb{R} = [-2, \frac{1}{4}]$ has small arithmetic capacity. In this note, we revisit their proof. We then determine the totally real parabolic parameters.

Proposition 2. *The only totally real parameters $c \in \overline{\mathbb{Q}}$ for which $z \mapsto z^2 + c$ has a parabolic cycle are $\frac{1}{4}$, $-\frac{3}{4}$, $-\frac{5}{4}$ and $-\frac{7}{4}$.*

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1. Postcritically finite parameters

We first revisit the proof of Noytaptim and Petsche [NP].

Proof of Proposition 1. Assume that c is a totally real postcritically finite parameter. Then, c and all of its Galois conjugates are real postcritically finite parameters, thus lie in the interval $[-2,0]$. Indeed,

- for $c \in (0, \frac{1}{4})$, the orbit of 0 under iteration f_c is infinite, converging to an attracting fixed point of f_c;
- for $c = \frac{1}{4}$ the orbit of 0 under iteration f_c is infinite, converging to a parabolic fixed point of f_c at $z = \frac{1}{2}$;
- for $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$ the orbit of 0 under iteration f_c is infinite, converging to ∞.

Let a be a solution of $a + 1/a = c$; that is, $a^2 - ca + 1 = 0$. Then a is an algebraic integer of modulus 1 with nonpositive real part, and all of its Galois conjugates also have modulus 1 and nonpositive real part.

By Kronecker's theorem, a is a root of unity. And since the Galois conjugates of a all have nonpositive real part, the only possibilities are the following:

- $a = -1$, which is mapped to $c = -2$;
- $a = \mathrm{e}^{\pm \mathrm{i}2\pi/3}$, which is mapped to $c = -1$;
- $a = \pm\mathrm{i}$, which is mapped to $c = 0$.

Therefore, the only postcritically finite parameters that are totally real are -2, -1, and 0. □

2. Parabolic parameters

We now present the proof of Proposition 2. Note that $c = \frac{1}{4}$, $c = -\frac{3}{4}$, $c = -\frac{5}{4}$ and $c = -\frac{7}{4}$ are indeed parabolic parameters. Indeed,

- $f_{\frac{1}{4}}$ has a fixed point with multiplier 1 at $z = \frac{1}{2}$;
- $f_{-\frac{3}{4}}$ has a fixed point with multiplier -1 at $z = -\frac{1}{2}$;
- $f_{-\frac{5}{4}}$ has a cycle of period 2 with multiplier -1 consisting of the two roots of $4z^2 + 4z - 1$;
- $f_{-\frac{7}{4}}$ has a cycle of period 3 with multiplier 1 consisting of the three roots of $8z^3 + 4z^2 - 18z - 1$.

Proof of Proposition 2. Assume that c is a totally real parabolic parameter. Then, the Galois conjugates of c also are parabolic parameters. Either $c = \frac{1}{4}$, $c = -\frac{3}{4}$, $c = -\frac{5}{4}$, or c and all of its Galois conjugates lie in the interval $[-2, -\frac{5}{4})$. Indeed, a parabolic cycle must attract the orbit of 0 under iteration of f_c. However,

- for $c \in (-\frac{3}{4}, \frac{1}{4})$, the orbit of 0 under iteration f_c converges to an attracting fixed point of f_c;
- for $c \in (-\frac{5}{4}, -\frac{3}{4})$ the orbit of 0 under iteration f_c converges to an attracting cycle of period 2 of f_c;
- for $c \in (-\infty, -2) \cup (\frac{1}{4}, +\infty)$ the orbit of 0 under iteration f_c converges to ∞.

Let us assume that $c \in [-2, -\frac{5}{4})$. Then, $b := 4c+6$ and all of its Galois conjugates lie in the interval $[-2, 1)$. Let a be a solution of $a + 1/a = b$; that is, $a^2 - ba + 1 = 0$.

Then a is an algebraic integer of modulus 1 with real part less than $\frac{1}{2}$, and all of its Galois conjugates also have modulus 1 and real part less than $\frac{1}{2}$.

By Kronecker's theorem, a is a root of unity. And since the Galois conjugates of a all have real part less than $\frac{1}{2}$, the only possibilities are the following:

- $a = -1$, $b = -2$ and $c = -2$; this is not a parabolic parameter;
- $a = \mathrm{e}^{\pm \mathrm{i}2\pi/3}$, $b = -1$ and $c = -\frac{7}{4}$; this is indeed a parabolic parameter;
- $a = \pm\mathrm{i}$, $b = 0$ and $c = -\frac{3}{2}$; in that case $4c = -6$ is not an algebraic unit in $\overline{\mathbb{Z}}/2\overline{\mathbb{Z}}$ and so, c is not a parabolic parameter;
- $a = \mathrm{e}^{\pm \mathrm{i}2\pi/5}$, $b = 2\cos(\frac{2\pi}{5})$ and $c = \frac{\sqrt{5}-13}{8}$; in this case, f_c has an attracting cycle of period 4 and so, c is not a parabolic parameter;
- $a = \mathrm{e}^{\pm \mathrm{i}4\pi/5}$, $b = 2\cos(\frac{4\pi}{5})$ and $c = \frac{-\sqrt{5}-13}{8}$; then the Galois conjugate $\frac{\sqrt{5}-13}{8}$ is not a parabolic parameter and so, c is not a parabolic parameter.

This completes the proof of the proposition. □

Remark: the following proof that $-\frac{3}{2}$ is not a parabolic parameter was explained to us by Valentin Huguin. It follows from [Bo] that for all $n \geq 1$,

$$\mathrm{discriminant}\big(f_c^{\circ n}(z) - z, z\big) = P_n(4c) \quad \text{with} \quad P_n(b) \in \mathbb{Z}[b] \quad \text{and} \quad \pm P_n \text{ monic.}$$

As an example,

$$P_1(b) = -b + 1, \quad P_2(b) = (b-1)(b+3)^3, \quad P_3(z) = (b-1)(b+7)^3(b^2+b+7)^4,$$

and

$$P_4(z) = (b-1)(b+3)^3(b+5)^6(b^3+9b^2+27b+135)^4(b^2-2b+5)^5.$$

Note that this yields an alternate proof that $c = \frac{1}{4}$, $c = -\frac{3}{4}$, $c = -\frac{5}{4}$ and $c = -\frac{7}{4}$ are parabolic parameters. In addition,

$$P_n(0) = \mathrm{discriminant}\big(z^{2^n} - z, z\big) \equiv 1 \pmod{2}.$$

As a consequence

$$P_n(-6) \equiv 1 \pmod{2}.$$

Thus, for all $n \geq 1$, the roots of $f_{-\frac{3}{2}}^{\circ n}(z) - z$ are simple, which shows that $f_{-\frac{3}{2}}$ has no parabolic cycle.

References

[Bo] T. Bousch, *Les racines des composantes hyperboliques de M sont des quarts d'entiers algébriques*, Frontiers in Complex Dynamics: a volume in honor of John Milnor's 80th birthday, A. Bonifant, M. Lyubich, S. Sutherland, editors. Princeton University Press 2014, 25–26.

[Bu] X. Buff, *On Postcritically Finite Unicritical Polynomials*, New York J. Math. 24 (2018) 1111–1122.

[M] J. Milnor, *Arithmetic of unicritical polynomial maps*, Frontiers in Complex Dynamics: a volume in honor of John Milnor's 80th birthday, A. Bonifant, M. Lyubich, S. Sutherland, editors. Princeton University Press 2014, 15–24.

[NP] C. Noytaptim & C. Petsche, *Totally real algebraic integers in short intervals, Jacobi polynomials, and unicritical families in arithmetic dynamics*, Preprint (2022).

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