REALIZING POLYNOMIAL PORTRAITS

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Abstract. It is well known that the dynamical behavior of a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is governed by the forward orbits of the critical points of f. The map f is said to be *postcritically finite* if every critical point has finite forward orbit, or equivalently, if every critical point eventually maps into a periodic cycle of f. We encode the orbits of the critical points of f with a finite directed graph called a *ramification portrait*. In this article, we study which graphs arise as ramification portraits. We prove that every abstract polynomial portrait is realized as the ramification portrait of a postcritically finite polynomial, and classify which abstract polynomial portraits can only be realized by unobstructed maps.

1. Introduction

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere, and let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. By the Riemann-Hurwitz formula, f has $2d-2$ critical points, counted with multiplicity; the critical set of f is denoted as C_f. The postcritical set of f, denoted P_f, is the smallest forward invariant subset of the Riemann sphere that contains the critical values of f. If the postcritical set is finite, the rational map is said to be *postcritically finite*. Associated to a postcritically finite rational map is a *ramification portrait*; that is, a finite directed graph Γ_f that encodes the action of f restricted to $C_f \cup P_f$. As an example, consider the polynomial $f : z \mapsto z^2 - 2$. The critical set is $C_f = \{0, \infty\}$, the postcritical set is $P_f = \{-2, 2, \infty\}$, and the ramification portrait is:

$$\infty \circlearrowleft 2 \qquad 0 \xrightarrow{2} -2 \longrightarrow 2 \circlearrowleft$$

In the portrait above, there is an edge from vertex x to vertex y if and only if $y = f(x)$. This edge is weighted with the positive integer $\deg_f(x)$, the local degree of f at x. To lighten notation, we record the weight of the edge from x to $f(x)$ if and only if $\deg_f(x) > 1$; that is, if and only if $x \in C_f$. The portrait above is a *polynomial portrait*; that is, there is a fixed vertex mapping to itself with full degree (the vertex ∞).

In this article, we study which graphs are isomorphic to portraits from postcritically finite polynomials. There are immediate necessary conditions

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that arise from local degree restrictions, and from Riemann-Hurwitz restrictions (see Section 2). We prove that in the polynomial setting, these conditions are also sufficient. A weighted finite directed graph as above which satisfies these conditions is called an *abstract polynomial portrait.*

Theorem 1. *Let Γ be an abstract polynomial portrait. Then there exists a polynomial $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $\Gamma_f \simeq \Gamma$.*

To prove Theorem 1, we construct an explicit *topological polynomial* $g : S^2 \to S^2$ so that $\Gamma_g \simeq \Gamma$. We build g so that it has no *obstructing multicurves.* It then follows from Thurston's Topological Characterization of Rational Maps that g is *combinatorially equivalent* to a polynomial f, so $\Gamma_f \simeq \Gamma_g$.

We cannot strengthen Theorem 1 by removing the hypothesis that Γ is a polynomial portrait because of the following two phenomena. The first is dynamical and related to Thurston's theorem. The second is nondynamical and related to the Hurwitz problem.

Portraits that can only be realized topologically. Consider the following abstract portrait Γ.

$$\begin{tikzcd} a \arrow[r, bend left, "2"] & b \arrow[l, bend left] & c \arrow[r, bend left, "2"] & d \arrow[l, bend left] \end{tikzcd}$$

Suppose there is a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $\Gamma \simeq \Gamma_f$. Then f is a quadratic rational map with two periodic cycles of period 2. However, a quick computation reveals that a quadratic rational map can have at most one periodic cycle of period 2, so no such f exists.

Even though no rational map f exists so that $\Gamma_f \simeq \Gamma$, it is possible to construct a topological branched cover $g : S^2 \to S^2$ so that $\Gamma_g \simeq \Gamma$. For example, after identifying S^2 with $\widehat{\mathbb{C}}$, we could take the squaring map $s : z \mapsto z^2$ and postcompose with an orientation-preserving homeomorphism $h : S^2 \to S^2$ so that $h(\infty) = 1, h(1) = \infty, h(0) = 2$, and $h(4) = 0$. Then $g := h \circ s$ is a branched cover with $\Gamma_g \simeq \Gamma$. By Thurston's theorem, the map g will necessarily admit an obstructing multicurve (see Section 2).

We are aware of a few methods to construct portraits that can only be realized topologically that are similar in spirit to the example above. It would be interesting to put these examples into a more general context.

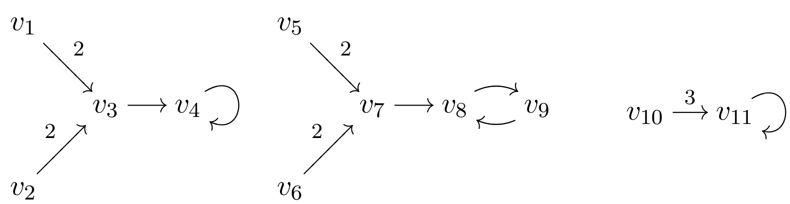
Question 2. Which abstract portraits Γ can only be realized topologically?

Portraits that cannot even be realized topologically. The *Hurwitz problem* is to characterize which *branch data* arise from branched covering maps $S^2 \to S^2$. See [1], [3], [6] and [8]. For example, it is known that there is no branched cover with the branch data $(2, 2), (2, 2), (3, 1)$. That is, there is no branched cover $f : S^2 \to S^2$ of degree 4 with exactly three critical values $\{v_1, v_2, v_3\} \subseteq S^2$, so that

- $f^{-1}(\{v_1\})$ contains exactly two points, each mapping forward with local degree two,

- $f^{-1}(\{v_2\})$ contains exactly two points, each mapping forward with local degree two, and
- $f^{-1}(\{v_3\})$ contains exactly two points, one mapping forward with local degree 3, and the other mapping forward with local degree 1.

This fact has dynamical consequences. Indeed, any abstract portrait Γ with this branch data cannot be the portrait of a branched covering map $S^2 \to S^2$, and therefore, Γ cannot be the portrait of a rational map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. For example, the following portrait has branch data $(2, 2), (2, 2), (3, 1)$.



While the general Hurwitz problem is unsolved, we note that all polynomial branch data are realizable (see Proposition 5.2 in [3]). We will not use this fact to construct the branched cover $g : S^2 \to S^2$ in the proof of Theorem 1.

Thurston's theorem. Let S^2 denote an oriented topological 2-sphere, and let $f\colon S^2 \to S^2$ be an orientation-preserving branched cover of degree $d \geq 2$ so that the postcritical set P_f is finite. We call such a map f a *Thurston map*. For convenience in stating the theorem, we assume that the orbifold of f is hyperbolic[1]. Two Thurston maps $f : (S^2, P_f) \to (S^2, P_f)$ and $g : (S^2, P_g) \to (S^2, P_g)$ are *combinatorially equivalent* provided that there are orientation-preserving homeomorphisms $\phi_0 : (S^2, P_f) \to (S^2, P_g)$ and $\phi_1 : (S^2, P_f) \to (S^2, P_g)$ so that

- $\phi_0 \circ f = g \circ \phi_1$, and
- the homeomorphisms ϕ_0 and ϕ_1 are isotopic relative to P_f.

In the 1980s, William Thurston proved that every Thurston map f is combinatorially equivalent to a rational map, or it is *obstructed*. In the latter case, f admits an invariant curve system called an *obstructing multicurve*.

A *multicurve* Δ is a finite collection of simple disjoint curves in $S^2 \setminus P_f$, no two of which are homotopic. All components $\delta \in \Delta$ are also required to be *essential* (δ does not bound a disk), and nonperipheral (δ does not bound a disk with exactly one puncture). The multicurve Δ is said to be *invariant* for f provided that for all $\delta \in \Delta$, every component of $f^{-1}(\{\delta\})$ is either

- homotopic to some $\delta' \in \Delta$ in $S^2 \setminus P_f$, or
- 'erased'; that is, it is peripheral or inessential.

Given an invariant multicurve Δ for f, Thurston defined an associated linear transformation $\mathbb{R}^\Delta \to \mathbb{R}^\Delta$ that encodes how different components of $f^{-1}(\Delta)$ map to Δ. The matrix for this transformation has non-negative real

[1] This condition essentially excludes power maps $z \mapsto z^n$, Chebyshev maps, and Lattès maps; see [2].

entries, so there is a leading eigenvalue λ which is real and non-negative. The multicurve Δ is an *obstruction* provided that $\lambda \geq 1$. If the Thurston map f admits an obstruction, f is said to be *obstructed*. If not, f is said to be *unobstructed*.

Theorem (Thurston's Topological Characterization of Rational Maps, [2])**.** *Let $f : (S^2, P_f) \to (S^2, P_f)$ be a Thurston map, and suppose that f has a hyperbolic orbifold. Then f is combinatorially equivalent to a rational map F if and only if f is unobstructed. In this case, F is unique up to conjugation by Möbius transformations.*

Levy cycles. For a given Thurston map $f : (S^2, P_f) \to (S^2, P_f)$, verifying the criterion in Thurston's theorem is difficult as it involves an infinite search in general. In this article, we will work with Thurston maps that are *topological polynomials*; that is, there is some $\omega \in S^2$ that is a fully ramified fixed point of f. More is known about Thurston's criterion in the case of topological polynomials.

A *Levy cycle* for the Thurston map $f : (S^2, P_f) \to (S^2, P_f)$ is a circularly ordered collection of simple closed curves $\{\delta_0, \ldots, \delta_{n-1}, \delta_n = \delta_0\}$ on $S^2 \setminus P_f$ such that

- no two curves are homotopic relative to P_f,
- the curves are pairwise disjoint,
- each curve is essential and nonperipheral, and
- for all $1 \leq i \leq n$, at least one component of $f^{-1}(\delta_i)$ is homotopic to δ_{i-1} and maps to δ_i by degree 1, where we take the indices modulo n.

Silvio Levy proved the following results in his thesis, [7].

Theorem (Levy)**.** *Let $f : (S^2, P_f) \to (S^2, P_f)$ be a Thurston map that is a topological polynomial. Then f is obstructed if and only if f admits a Levy cycle.*

Theorem (Levy)**.** *Let Γ be an abstract polynomial portrait such that every critical vertex is periodic. Then every Thurston map realizing Γ is unobstructed.*

The proof of the latter can be strengthened to give the following result. See, for example, Hubbard [4, Theorem 10.3.9].

Theorem (Levy-Berstein)**.** *Suppose Γ is an abstract polynomial portrait such that each cycle contains a critical vertex. Then every Thurston map realizing Γ is unobstructed.*

We will use Levy's first theorem in an essential way in our proof of Theorem 1. Indeed, given an abstract portrait Γ, we will construct a topological polynomial $g : S^2 \to S^2$ so that $\Gamma_g \simeq \Gamma$, and so that g cannot possibly admit a Levy cycle. Theorem 1 immediately follows.

In [5, Theorem 1.1] Kelsey uses self-similar groups to give a partial converse to the Levy-Berstein theorem. In the discussion that follows, an *attractor* of an abstract portrait is a cycle that contains a critical vertex, and a *non-attractor* is a cycle that does not contain a critical vertex.

Theorem (Kelsey)**.** *Suppose* Γ *is an abstract polynomial portrait, and that* Γ *satisfies at least one of the following properties:*

(1) Γ *contains a cycle*[2] *of length at least two that does not contain any critical vertices;*
(2) Γ *contains at least two cycles*[3] *that do not contain any critical vertices;*
(3) Γ *contains at least two non-attractor cycles that have length at least two;*
(4) Γ *contains at least four non-attractor cycles.*

Then there is an obstructed Thurston map that realizes Γ*.*

In Theorem 3, we show that certain abstract polynomial portraits have only unobstructed representatives, and in Theorem 4, we show that certain abstract polynomial portraits have obstructed representatives. We need a definition to state Theorems 3 and 4. Let Γ be an abstract polynomial portrait, and let v be a vertex of Γ. Then v is the source vertex of exactly one edge of Γ. We let $\tau(v)$ denote the target vertex of this edge.

Theorem 3. *Suppose* Γ *is an abstract polynomial portrait that has at least four postcritical vertices and satisfies one of the following properties.*

(i) Γ *has a single non-attractor cycle, and it has length one.*
(ii) *Every finite postcritical vertex of* Γ *is in a single non-attractor cycle, this cycle has length* p^k *for some prime number* p *and some positive integer* k*, and the finite postcritical vertices can be enumerated as* $\{v_i : 0 \leq i < p^k\}$ *such that* $\tau(v_i) = v_{i+1}$ *(mod* p^k*) for every* $i \in \{0, \ldots, p^k - 1\}$*, and if* v_j *is a critical value then* j *is a multiple of* p^{k-1}*.*

Then every Thurston map with portrait isomorphic to Γ *is unobsructed.*

The hypothesis that there are at least four postcritical vertices is not restrictive, since by Thurston's characterization theorem a Thurston map with fewer than four postcritical points is unobstructed. The proof is along the lines of the argument for the Levy-Berstein Theorem. If the abstract portrait can be realized by an obstructed Thurston map, then by Levy [7] there must be a Levy cycle. This implies that, in the teminology of Hubbard [4], there must be a *degenerate Levy cycle*. One then shows that this is impossible if the portrait satisfies (i) or (ii). The proof is given in Section 4. Part (if not all) of case (i) of Theorem 3 was previously known. The case

[2]which is necessarily a non-attractor
[3]which are necessarily non-attractors

of a single non-attractor cycle of length one and no other finite cycles was observed by Kelsey [5, p. 52].

Theorem 4. *Suppose Γ is an abstract polynomial portrait that has at least four postcritical vertices and satisfies one of the following properties.*

(i) *Every finite postcritical vertex of Γ is in a single non-attractor cycle, this cycle has length p^k for some prime number p and some positive integer k, the finite vertices can be enumerated as $\{v_i : 0 \leq i < p^k\}$ such that v_0 is a critical value, $\tau(v_i) = v_{i+1}$ mod p^k for every $i \in \{0, \dots, p^k - 1\}$, and there is a critical value v_j such that j is not a multiple of p^{k-1}.*

(ii) *Every finite postcritical vertex of Γ is in a single non-attractor cycle of length at least two, and this cycle does not have prime-power length.*

(iii) *Γ contains a non-attractor cycle of length at least two that does not contain all of the finite critical values.*

(iv) *Γ has at least two non-attractor cycles of length one.*

Then there exists an obstructed Thurston map whose portrait is isomorphic to Γ.

The proof of Theorem 4 is constructive and relies on a combinatorial lemma, Lemma 7. Given an abstract polynomial portrait Γ that satisfies any of conditions (i)-(iv) of the theorem, we describe a construction of an obstructed Thurston map with portrait isomorphic to Γ. We introduce *rose maps* and prove the lemma in Section 5. We then prove the theorem in Section 6.

Combining Theorem 3, Theorem 4, and the Levy-Berstein Theorem, we classify the abstract polynomial portraits that are completely unobstructed. We summarize this result in the following theorem, which we prove in Section 7.

Theorem 5. *Suppose Γ is an abstract polynomial portrait. Then every Thurston map with portrait isomorphic to Γ is unobstructed if and only if Γ satisfies at least one of the following conditions.*

(i) *Γ has at most three postcritical vertices.*

(ii) *Every cycle of Γ is an attractor.*

(iii) *Γ has a single non-attractor cycle, and it has length one.*

(iv) *Every finite postcritical vertex of Γ is in a single non-attractor cycle, this cycle has length p^k for some prime number p and some positive integer k, the finite postcritical vertices can be enumerated as*

$$\{v_i : 0 \leq i < p^k\} \text{ such that } \tau(v_i) = v_{i+1} \bmod p^k \text{ for every } i \in \{0, \dots, p^k - 1\},$$

and if v_j is a critical value, then j is a multiple of p^{k-1}.

2. Preliminaries

Portraits associated to Thurston maps. Let $f : (S^2, P_f) \to (S^2, P_f)$ be a Thurston map of degree d. The *ramification portrait* of f is the weighted directed graph Γ such that the vertex set $V(\Gamma)$ is the union of the set C_f of critical points and the set P_f of postcritical points, and for each vertex v there is an edge from v to $f(v)$ with weight the local degree $\deg_f(v)$ of f at v. By the Riemann-Hurwitz formula,

$$\sum_{v \in C_f} (\deg_f(v) - 1) = 2d - 2.$$

Since f has degree d, at each vertex v the sum of the weights of the incoming edges is at most d. Note that f is a topological polynomial if and only if there is a vertex v such that $f(v) = v$ and $\deg_f(v) = d$.

Abstract portraits. Suppose Γ is a finite weighted directed graph (with the weights positive integers) such that each vertex of Γ is the source of exactly one edge. Let $\tau\colon V(\Gamma) \to V(\Gamma)$ be the function which takes a vertex v to the target of the edge with source v. We call the weight of the edge from v to $\tau(v)$ the *degree* of τ at v and denote it by $\deg(v)$. A vertex v is *critical* if $\deg(v) > 1$, and is *postcritical* if there are a critical vertex w and a positive integer k such that $\tau^{\circ k}(w) = v$. If v is a critical vertex, then $\tau(v)$ is called a *critical value*. We denote the set of critical vertices by C_Γ, and we denote the set of postcritical vertices by P_Γ. We say that Γ is an *abstract portrait* if it satisfies the following:

- every vertex of Γ is either critical or postcritical,
- there is an integer $d \geq 2$ such that $\sum_{v \in C_\Gamma} (\deg(v) - 1) = 2d - 2$, and
- for each vertex v the sum of the weights of the edges with target v is at most d.

We call d the *degree* of the abstract portrait. We say that an abstract portrait Γ is *realized* by a Thurston map f if Γ is isomorphic to the portrait of f (as weighted directed graphs). An abstract portrait Γ is *realizable* if it is realized by some Thurston map.

An abstract portrait of degree d is an *abstract polynomial portrait* if there is a vertex v such that $\tau(v) = v$ and $\deg(v) = d$. In this case we choose such a vertex and call it ∞; the other vertices are called *finite*. We call a cycle (of the action of τ on $V(\Gamma)$) *finite* if all of its vertices are finite; that is, a cycle is finite if it does not consist of the singleton ∞.

Finite subdivision rules. We define finite subdivision rules in the present context of Thurston maps. A finite subdivision rule $\mathcal{R}$ consists of the structure $S_\mathcal{R}$ of a finite CW complex on the 2-sphere (called the model subdivision complex), a subdivision $\mathcal{R}(S_\mathcal{R})$ of $S_\mathcal{R}$ and a continuous cellular map $\sigma_\mathcal{R}\colon \mathcal{R}(S_\mathcal{R}) \to S_\mathcal{R}$ (called the *subdivision map*) whose restriction to each open cell is a homeomorphism onto an open cell. Furthermore, for each closed 2-cell $\widetilde{t}$ of $S_\mathcal{R}$ there are (i) a cell structure t (called the *tile type* of $\widetilde{t}$)

on the 2-disk D^2 such that the 1-skeleton of t is ∂D^2 and (ii) a continuous surjection $\psi_t\colon\, t \to \widetilde{t}$ (called the *characteristic map* of $\widetilde{t}$) whose restriction to each open cell is a homeomorphism onto an open cell.

The map $\sigma_{\mathcal{R}}$ is a Thurston map if it has degree at least 2. Conversely, a Thurston map f is the subdivision map of a finite subdivision rule if and only if there exists a connected finite f-invariant graph G which contains the postcritical set of f. Such a graph G serves as the 1-skeleton of a model subdivision complex.

3. Realizing a portrait by an unobstructed map

In this section we prove Theorem 1. We begin with an example to illustrate the construction. Consider the abstract portrait Γ that is shown below.

$$\begin{tikzcd}
* \arrow[r, "2"] & a \arrow[r] & b \arrow[r, bend right] & c \arrow[l, bend right] & e \arrow[r, bend right, "2"'] & f \arrow[l, bend right] & \infty \arrow[loop right, "4"] \\
& * \arrow[r, "2"] & d \arrow[ur] & & & &
\end{tikzcd}$$

The proof defines an ordering of the finite postcritical vertices of Γ. In this case we use the ordering given by $a < b < c < d < e < f$. Following the terminology that will be defined in the proof, the ordered sets (a, b, c) and (d) are called type-1 chains and the ordered set (e, f) is called a type-2 chain. (The first element of a type-1 chain is the image of a critical vertex that is not postcritical, and the first element of a type-2 chain is a periodic critical vertex.) The model subdivision complex $S_{\mathcal{R}}$ is shown in Figure 1 as a stereographic projection of S^2 to the plane. The 1-skeleton will always be a star graph with central vertex ∞. The vertices of Γ are identified with the vertices of $S_{\mathcal{R}}$. The ordering of the finite postcritical vertices chosen above determines the counterclockwise ordering of the labels of the vertices in Figure 1. The tile type t is shown in Figure 2; $S_{\mathcal{R}}$ is the image of t under the characteristic map $\psi\colon\, t \to S_{\mathcal{R}}$. The *label* of a vertex v of t is $\psi(v)$; if $\psi(v) \neq \infty$ then v is called a *finite* vertex.

We will give a combinatorial description of the subdivision $\mathcal{R}(S_{\mathcal{R}})$. We first add edges to $S_{\mathcal{R}}$ that will ensure that the subdivision map cannot have any Levy cycles (stage 1), and then add more edges to get $\mathcal{R}(S_{\mathcal{R}})$ (stages 2 and 3). Figure 3 shows the construction after the first stage from the point of view of the tile type t. No further changes are made in the second stage since there are already the correct number of subtiles. The *label* of each vertex is drawn outside t. Every vertex of the subdivision whose label is not ∞ is a *finite* vertex. Every finite vertex v has an *image label*, which is $\sigma_{\mathcal{R}}(\psi(v))$. It is drawn inside t. (Of course, this is abuse of notation, since we haven't finished the construction yet and hence haven't defined the subdivision map $\sigma_{\mathcal{R}}$ yet.)

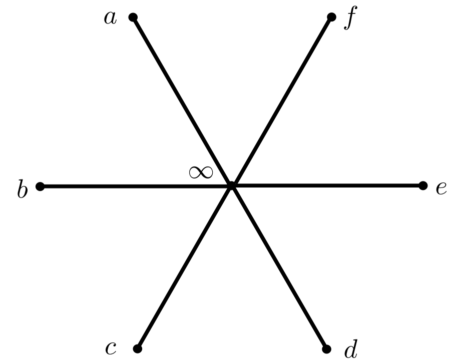


FIGURE 1. The model subdivision complex $S_{\mathcal{R}}$

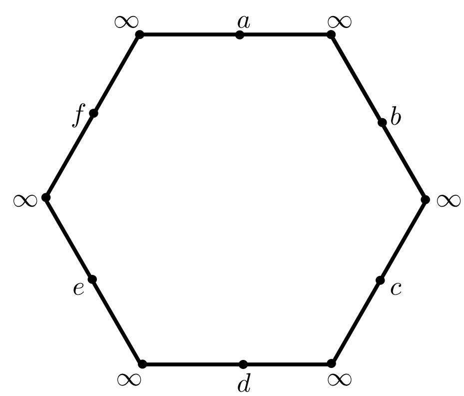


FIGURE 2. The tile type t, which ψ maps to $S_{\mathcal{R}}$ by identifying edges in pairwise fashion

To complete the construction (stage 3) we add stickers as needed in each subtile so that each subtile is a 12-gon, every other vertex is the original vertex labeled ∞, and the image labels of its finite vertices are in the proper cyclic order. (A sticker is an edge with a vertex of valence one, resembling a stick pin with a spherical head.) It is straightforward to define the subdivision map $\sigma_{\mathcal{R}}$ so that its restriction to each open cell is a homeomorphism to an open cell and it takes each finite vertex to its image label. Figure 4 shows the subdivision of the tile type t, and Figure 5 shows the subdivision $\mathcal{R}(S_{\mathcal{R}})$.

If γ is a simple closed curve in $S^2\backslash P$, let D_γ be the component of $S^2\backslash\gamma$ that does not contain ∞. If γ is an element of a Levy cycle (or, more generally, of a multicurve), then D_γ must contain at least two postcritical points. The five new edges in Figure 3 ensure that if we extend the subtiling so that it combinatorially describes a finite subdivision rule, then the subdivision map cannot have a Levy cycle. This can be proven as follows. In the model subdivision complex, the new arc whose barycenter has label d bounds a closed disk D such that $\operatorname{int}(D) \cap P_f = \{c\}$ and its boundary contains ∞. It follows from Lemma 6 that for any positive integer n, each element of a Levy cycle can be isotoped rel the postcritical set to be disjoint from all new edges of the n^{th} subdivision $\mathcal{R}^n(S_{\mathcal{R}})$. Since the interior of the disk D contains the single postcritical point c and its boundary contains ∞, the vertex c cannot be in the open disk D_γ for a Levy curve γ. In the next two subdivisions there will be new edges enclosing the stickers with vertices d, b and a, so none of these vertices could be in the open disk D_γ for a Levy curve γ. There is a new edge joining the vertex e to an ∞-vertex, so the vertex e cannot be in the open disk D_γ for a Levy curve γ. In the next subdivision there will be a new edge joining the vertex labeled f to an ∞-vertex, so that vertex cannot be in the open disk D_γ for a Levy curve γ. Hence no finite vertex can be in a Levy disk, so there are no Levy cycles and hence the subdivision map is equivalent to a rational map. This concludes our example.

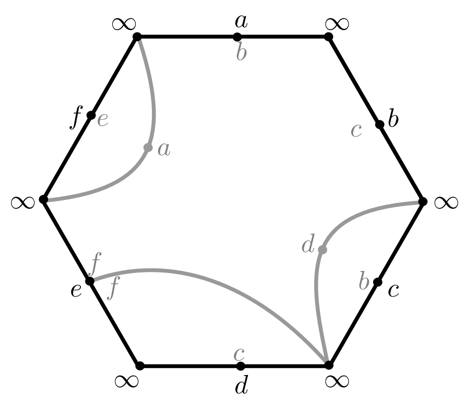


FIGURE 3. The construction after stage 1

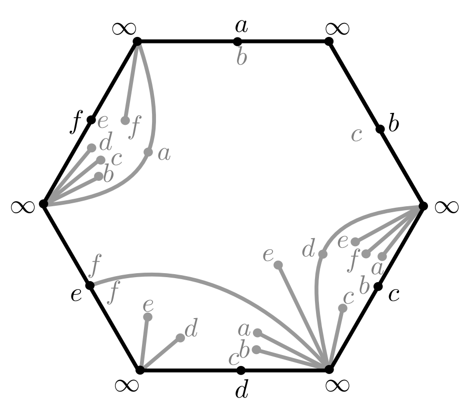


FIGURE 4. The subdivision of the tile type after stage 3

We call an edge of a subdivision $\mathcal{R}^n(S_{\mathcal{R}})$ (of a finite subdivision rule $\mathcal{R}$) a *new edge* if it is not contained in an edge of $S_{\mathcal{R}}$. The following lemma plays a crucial role in the proof of Theorem 1.

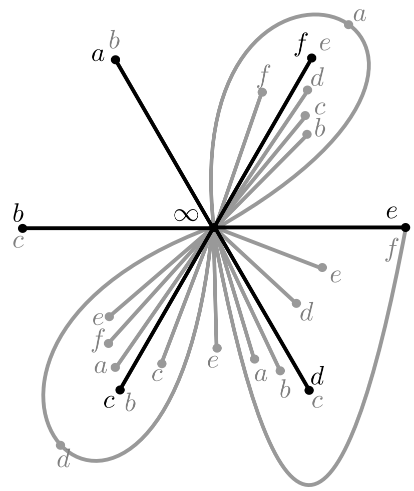


FIGURE 5. The subdivision of the model subdivision complex

Lemma 6. *Suppose f is a Thurston map which is also the subdivision map $\sigma_{\mathcal{R}}$ of a finite subdivision rule $\mathcal{R}$. Suppose $\{\delta_0, \dots, \delta_{k-1}, \delta_k = \delta_0\}$ is a Levy cycle for f and let n be a positive integer. Then for each $i \in \{1, \dots, k\}$, δ_i can be isotoped rel P_f so that it is disjoint from each new edge of the subdivision $\mathcal{R}^n(S_{\mathcal{R}})$.*

Proof. We first assume $n = 1$. Let E be the 1-skeleton of $S_{\mathcal{R}}$ and let E_1 be the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$. Each δ_i can be isotoped so that $\delta_i \cap E_1$ is finite. For each $i \in \{0, \dots, k\}$, let a_i be the minimum of $\#(\delta \cap E)$, where δ is a curve that is isotopic rel P_f to δ_i, and let b_i be the minimum of $\#(\delta \cap E_1)$, where δ is a curve that is isotopic rel P_f to δ_i. Let $i \in \{1, \dots, k\}$, and let δ be a curve that is isotopic rel P_f to δ_i such that $a_i = \#(\delta \cap E)$. Let γ be a component of $f^{-1}(\delta)$ which maps to δ by degree 1 and is isotopic rel P_f to δ_{i-1}. Then $a_{i-1} \le b_{i-1} \le \#(\gamma \cap E_1) = a_i$. Since this is true for every i and i varies cyclically, each of these inequalities is an equality. So $a_{i-1} = a_i$ and $a_{i-1} = b_{i-1}$. This implies that γ doesn't intersect $E_1 \setminus E$. This establishes the result for $n = 1$.

Now suppose that $n > 1$. Let p be a positive integer with $p \ge n$ and $p \equiv 1 \mod k$. Then $\{\delta_0, \dots, \delta_{k-1}, \delta_k = \delta_0\}$ is a Levy cycle for $f^{\circ p}$. By the previous paragraph applied to $f^{\circ p}$, each δ_i can be isotoped so that it does not intersect any new edge of $\mathcal{R}^p(S_{\mathcal{R}})$. Since each new edge of $\mathcal{R}^n(S_{\mathcal{R}})$ is a union of new edges of $\mathcal{R}^p(S_{\mathcal{R}})$, then each δ_i can be isotoped so that it is disjoint from each new edge of $\mathcal{R}^n(S_{\mathcal{R}})$. □

Proof of Theorem 1. Let Γ be an abstract polynomial portrait. Let $C' = C_\Gamma \setminus \{\infty\}$ (the set of finite critical vertices), and let $P' = P_\Gamma \setminus \{\infty\}$ (the set of finite postcritical vertices). Let $V_\Gamma = \{\tau(x) : x \in C_\Gamma\}$ (the set of critical values) and let $V' = V_\Gamma \setminus \{\infty\}$ (the set of finite critical values). Let

$A = \{v \in V' : v = \tau(c) \text{ for some } c \in C_\Gamma \setminus P_\Gamma\}$. For each $v \in A$, we choose an element $c_v \in C_\Gamma \setminus P_\Gamma$ with $\tau(c_v) = v$.

Let n be the cardinality of P'. A key step is to appropriately order the elements of P' by naming them $a_1, \dots, a_n$. To do this, we partition P' into *chains*. We define the chains recursively. We will put postcritical vertices that have already been placed in chains in a set $\widetilde{A}$. To begin the construction, let $\widetilde{A} = \emptyset$ and let $i = 1$.

The ordering. Suppose for the recursive step that $A \not\subset \widetilde{A}$, $i \in \{1, \dots, n\}$, and that we have already defined a_j for $j \in \{1, \dots, i-1\}$. If there is a vertex in $P' \setminus \widetilde{A}$ that is not periodic under τ, then we can choose an element $v \in A \setminus \widetilde{A}$ such that v is not the image under τ of a postcritical vertex. If every vertex in $P' \setminus \widetilde{A}$ is periodic under τ, choose $v \in A \setminus \widetilde{A}$. In each case, let $a_i = v$ and add v to $\widetilde{A}$. If $\tau(v) \notin \widetilde{A}$, we let $a_{i+1} = \tau(v)$ and add $\tau(v)$ to $\widetilde{A}$. We continue until we reach an index j such that $\tau(a_j)$ is in $\widetilde{A}$. At this point we stop this iteration of the recursion. We define the ordered set $(a_i, \dots, a_j)$ to be the *chain* of each of its elements. We call it a *type*-1 chain. It begins with an element of A. The *first element* of the chain is a_i, and the *last* element of the chain is a_j. The *length* of the chain is $j + 1 - i$. After redefining i to be $j + 1$, we continue this recursive step as long as possible.

Once we can no longer continue this recursion, the elements of P' which remain are exactly the elements of finite (attractor) cycles which are connected components of Γ. To start the next recursion, we choose a critical vertex v in a remaining attractor cycle and let $a_i = v$. Let k be the number of elements in the attractor cycle. For $1 \leq j < k$, let $a_{i+j} = \tau^{\circ j}(a_i)$. As before a_i is the *first* element of the chain, a_{i+k-1} is the *last* element of the chain, and the *length* of the chain is k. We call it a *type*-2 chain. We continue recursively to choose all of the points in the other attractor cycles. After doing this, the elements of P' are $a_1, \dots, a_n$ in order.

Construction of $S_\mathcal{R}$. We next construct the associated finite subdivision rule $\mathcal{R}$. The 1-skeleton of the model subdivision complex $S_\mathcal{R}$ is a tree as in Figure 1. There is one central vertex. We identify $\infty \in \Gamma$ with this central vertex. There are n "stickers" (a sticker is an edge of the graph with a vertex of valence one, like a stick pin with a spherical head) from ∞ to valence 1 vertices $a_1, \dots, a_n$, in counterclockwise order. We identify $a_1, \dots, a_n \in \Gamma$ with these valence 1 vertices. The tile type t is a $(2n)$-gon, which we think of as an n-gon with each edge bisected. The characteristic map $\psi : t \to S_\mathcal{R}$ maps the edge barycenters to the sticker heads and the other vertices to ∞. The edge barycenters are called finite vertices and the others are called ∞-vertices. More generally, a vertex of some subdivision of t which is not a vertex of t is called a finite vertex. Every vertex v of t is labeled by $\psi(v)$. These vertex labels are placed outside t in the figures. We use clockwise order on ∂t.

If k is a integer with $k \geq 2$, a *k-doodle* is a graph with three vertices and k edges (none of them loops) such that one vertex (the central vertex) has valence k, one vertex (the head) has valence 1, and the third vertex (the foot) has valence $k-1$. Note that a 2-doodle is a bisected arc.

We define a subdivision $\mathcal{R}(t)$ of t. We do this in three stages. We first define a subtiling of t into subtiles such that the finite vertices in each subtile are in the proper cyclic order. This means that there might be fewer than n of them, but they will have image labels, which are distinct elements of $\{a_1, ..., a_n\}$, and, when taken in clockwise order, their image labels have the same cyclic order as in $(a_1, ..., a_n)$. We then add arcs and k-doodles as determined by the critical vertices of Γ so that we have d subtiles. Finally, we add stickers as necessary to get the subdivision $\mathcal{R}(t)$. The tiles of the first stage will be defined so that the resulting subdivision map does not have Levy cycles. We do this by ensuring that there is an iterated subdivison $\mathcal{R}^n(t)$ of t such that for each finite vertex v of t except possibly one, either there is a new edge from v to an ∞-vertex or there is an arc (made out of two or four new edges) from the ∞-vertex before v to the ∞-vertex after v.

As we construct $\mathcal{R}(t)$, we will give *image labels* to its vertices. The image label of a vertex is the vertex it will map to under the analog of ψ from $\mathcal{R}(t)$ to $S_{\mathcal{R}}$. So for a vertex v in t, the image label of v is defined to be $\tau(\psi(v))$. We will keep track of the critical vertices that have already been accounted for during the construction of $\mathcal{R}(t)$ in a set $\widetilde{C}$. For the beginning of the construction, we define $\widetilde{C} = \emptyset$.

Stage 1. Suppose a_i is the last element of a chain, and a_j is the first element of the next chain (in cyclic order). So either $1 \leq i < n$ and $j = i + 1$ or $i = n$ and $j = 1$. The construction in stage 1 depends on the types of the chains which contain a_i and a_j. We consider various cases.

If a_i and a_j are in distinct chains of type 1 or if they are in the same chain of type 1 (there is only one chain) and $\tau(a_i) \neq a_j$, then we add a k-doodle, with k being the degree of the critical vertex c_{a_j}, with head the ∞-vertex after a_i and with foot the ∞-vertex before a_i. We give the central vertex of the k-doodle image label a_j, and add c_{a_j} to $\widetilde{C}$. See Figure 6, which, like Figures 7–10, is drawn with $k = 2$.

If a_i is in a chain of type 1 and a_j is in a chain of type 2, then we add $k-1$ edges joining a_j to the ∞-vertex before a_i (where $k = \deg(a_j)$) and add a_j to $\widetilde{C}$. See Figure 7.

Suppose a_i is in a chain of type 2 and a_j is in a chain of type 1 (this can only occur if $i = n$ and $j = 1$). If a_i is in a chain of length 1, then we don't do anything at this stage. If a_i is in a chain of length greater than 1, then we add a k-doodle (with $k = \deg(c_{a_j})$) with head the ∞-vertex after a_i and with foot the ∞-vertex before a_i. We give the central vertex of the k-doodle image label a_j, and add c_{a_j} to $\widetilde{C}$. Figure 8 shows both possibilities.

If a_i and a_j are both in chains of type 2 and a_i is in a chain of length 1, then to a_j we add $k-1$ edges joining it to the ∞-vertex before a_j (where

$k = \deg(a_j)$) and add a_j to $\widetilde{C}$. If a_i and a_j are both in chains of type 2 and a_i is in a chain of length greater than 1, then to a_j we add $k-1$ edges joining it to the ∞-vertex before a_i (where $k = \deg(a_j)$) and add a_j to $\widetilde{C}$. The two possibilities are shown in Figure 9.

Now suppose that there is a single chain, it has type 1, and $\tau(a_n) = a_1$. This is the only remaining case. The Riemann-Hurwitz condition implies that if there is just one finite critical value, then Γ has only two critical vertices and their degrees both equal the degree of Γ. Hence the finite critical vertex is the only vertex of Γ which τ maps to the finite critical value. This is impossible in the present case because $a_1 \in A$ and $\tau(a_n) = a_1$. So either one of the a_i's is a critical vertex or one of the a_i's with $i > 1$ is a critical value.

First suppose that $r \in \{1, \dots, n\}$ and a_r is a critical vertex with degree k. We add $k-1$ arcs in t from a_r to the ∞-vertex before a_r, and we add a_r to $\widetilde{C}$. See the left side of Figure 10.

If none of the a_i's is a critical vertex, then some a_i with $i > 1$ is a critical value. In this case, suppose $r \in \{2, \dots, n\}$ such that a_r is a critical value. Let $k_1 = \deg(c_{a_1})$, and let $k_r = \deg(c_{a_r})$. We add a k_r-doodle with head the ∞-vertex after a_n and with foot the ∞-vertex before a_n. We give its central vertex image label a_r, and we add c_{a_r} to $\widetilde{C}$. We then add a k_1-doodle with head the ∞-vertex after a_n and with foot the ∞-vertex before a_n as indicated in Figure 10. We give its central vertex image label a_1, and we add c_{a_1} to $\widetilde{C}$. See the right side of Figure 10. This completes stage 1 of the construction.

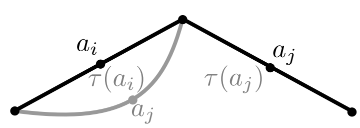


FIGURE 6. One or two chains of type 1

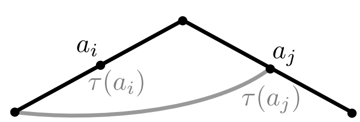


FIGURE 7. A chain of type 1 followed by a chain of type 2

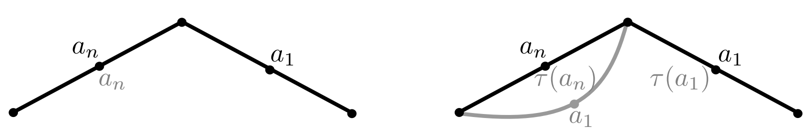


FIGURE 8. A chain of type 2 followed by a chain of type 1

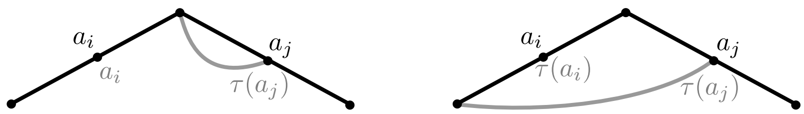


FIGURE 9. Two chains of type 2

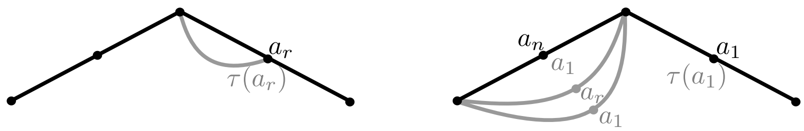


FIGURE 10. A single chain and it has type 1

Verification that image labels are consistent after Stage 1. We now look at what we have after stage 1. Every subtile except for the central one is either a 2-gon or a 4-gon, and so there are only one or two finite vertices. For a 2-gon there is only one finite vertex and so its image label is in proper cyclic order.

For a 4-gon, there are two finite vertices, so their image labels are in proper cyclic order if they are distinct. The only potential problem is if, in the notation of Figure 6, $\tau(a_i) = a_j$. Suppose that this happens. Then a_i and a_j are in different chains. Because $\tau(a_i) = a_j$ and a_i and a_j are in different chains, a_j is not periodic under τ. But if there exists such a vertex when a chain is defined, then the first vertex of that chain must not be the image of a postcritical vertex. So it is not possible that $\tau(a_i) = a_j$. Hence the two finite vertices of every 4-gon have different image labels.

Now we verify that the same is true for the central tile s. Suppose that $a_i, ..., a_k$ are the vertices of a chain in order. Then $a_i, ..., a_k$ are labels of consecutive finite vertices $v_i, ..., v_k$ of t. Moreover, $v_i, ..., v_{k-1}$ are consecutive finite vertices of s. Their image labels are $a_{i+1}, ..., a_k$. In the cases corresponding to Figure 6, Figure 10 and the right half of Figure 8, the finite vertex of s preceeding v_i has image label a_i. These are the only cases in which a_i is an image label of a vertex of s. In the situation of Figure 10, there is only one chain and $\tau(a_n) = a_1$, so it is clear in this case that the image labels of s are in proper cyclic order. In all other cases except those corresponding to the left halves of Figures 8 and 9, $\tau(a_k)$ is not the image label of a vertex of s. In the left halves of Figures 8 and 9, we have that $k = 1$ and $\tau(a_k) = a_k$. Hence the vertices among $a_i, ..., a_k$ which are image labels of vertices of s occur consecutively and in proper order. Finally, it is clear that the chains occur in proper order. So in the central tile the image labels of the finite vertices are in proper cyclic order.

Stage 2. For the second stage, we add subtiles corresponding to the critical vertices in C' that aren't in $\widetilde{C}$. We do this recursively. Each time we add subtiles because of an element c of $C' \setminus \widetilde{C}$, we add this element to $\widetilde{C}$. Since C' is finite, this process will terminate. Suppose $c \in C' \setminus \widetilde{C}$. Let $k = \deg(c)$. If

$c \in P'$, then we add $k-1$ edges from c to the ∞-vertex of t before c, and we add c to $\widetilde{C}$. Image labels of finite vertices of all tiles remain in proper cyclic order. Now suppose $c \notin P'$, but that there is another element $c' \in C' \setminus \widetilde{C}$ with $c' \notin P'$ and $\tau(c') \neq \tau(c)$. Let $k = \deg(c)$ and let $k' = \deg(c')$. Let $i, j \in \{1, \dots, n\}$ such that $\tau(c) = a_i$ and $\tau(c') = a_j$. Choose a subtile s of t. If s doesn't contain a vertex with image label a_i, then there is a unique ∞-vertex in s such that we can add a k-doodle with head and tail this ∞-vertex and with image label a_i and still have the image labels be in cyclic order. We do this, and we add c to $\widetilde{C}$. Image labels of finite vertices of all tiles remain in proper cyclic order. Suppose s does contain a vertex v with image label a_i. Then we add a k'-doodle to s with head the ∞-vertex after v, with central vertex with image label a_j, and with tail the ∞-vertex before v. We then add a k-doodle to s with head the ∞-vertex after v, with central vertex with image label a_i, and with tail the ∞-vertex before v. There are a new subtile in s with the same image labels (and in the same cyclic order) as for s, some 2-gons if $k > 2$ or $k' > 2$, and two 4-gons with a vertex labeled a_i and a vertex labeled a_j. So we still have the image labels of the finite vertices of all of the subtiles in cyclic order. Finally, we add c and c' to $\widetilde{C}$.

To complete stage 2, we need to consider the case that $C' \setminus \widetilde{C} \neq \emptyset$ and that all elements of $C' \setminus \widetilde{C}$ have the same image a_i under τ. Choose an element $c \in C' \setminus \widetilde{C}$, and let $k = \deg(c)$. Let $r = \#(C' \setminus \widetilde{C})$ and let m be the sum of the degrees of the elements of $C' \setminus \widetilde{C}$. At every step of the construction thus far, the number of subtiles of t increases by $\deg(v) - 1$, where v is the vertex added to $\widetilde{C}$. So the number of subtiles of t created thus far is

$$\begin{aligned} 1 + \Sigma_{v \in \widetilde{C}}(deg(v) - 1) &= 1 + \Sigma_{v \in C'}(deg(v) - 1) - m + r \\ &= 1 + d - 1 - m + r \\ &= (d - m) + r. \end{aligned}$$

Because τ maps every element of $C' \setminus \widetilde{C}$ to a_i, the number of subtiles that can have a vertex with image label a_i is at most $d - m$, so there is a subtile that does not have a vertex with image label a_i. There is a unique ∞-vertex in this subtile such that we can add a k-doodle with head and tail this ∞-vertex and with image label a_i and still have the image labels be in cyclic order. We do this, and we add c to $\widetilde{C}$. This completes the recursive step, so we can continue the recursion until $C' = \widetilde{C}$. This completes the second stage. At this point there are d subtiles of t, and in each subtile the image labels of the finite vertices are in proper cyclic order.

Stage 3. Suppose s is a subtile of the construction after stage two, and $i \in \{1, \dots, n\}$. If s doesn't have a vertex with image label a_i, then there is a unique ∞-vertex of s to which we can add a sticker whose other vertex has image label a_i and still have the image labels of the finite vertices in cyclic order. We do this for every such s and i. This completes stage 3.

Completion of the construction of $S_{\mathcal{R}}$. At this point every subtile has n finite vertices and their image labels are in proper cyclic order. We define this to be the subdivision $\mathcal{R}(t)$, and we define its image under the characteristic map $t \to S_{\mathcal{R}}$ to be $\mathcal{R}(S_{\mathcal{R}})$. It is straightforward to define a subdivision map that takes $\mathcal{R}(S_{\mathcal{R}})$ to $S_{\mathcal{R}}$, which takes each open cell homeomorphically to an open cell, takes each ∞-vertex to ∞, and takes a vertex with image label a_i to the vertex a_i. This completes the definition of the finite subdivision rule $S_{\mathcal{R}}$. It is clear from this construction that the ramification portrait of the subdivision map $\sigma_{\mathcal{R}}$ is isomorphic to Γ.

Verification that $\sigma_{\mathcal{R}}$ has no Levy cycle. We prove by contradiction that $\sigma_{\mathcal{R}}$ cannot have a Levy cycle. Suppose $\{\delta_0, \dots, \delta_{k-1}, \delta_k = \delta_0\}$ is a Levy cycle for $\sigma_{\mathcal{R}}$. Choose any $i \in \{1, \dots, k\}$, and consider the component D_i of $S^2 \setminus \delta_i$ that does not contain ∞. Since δ_i is essential and is not peripheral, D_i must contain at least two points of P. We will obtain a contradiction by showing that D_i can contain at most one point of P. For this, consider a finite postcritical point a_j of $\sigma_{\mathcal{R}}$ in $S_{\mathcal{R}}$.

Suppose that there exists a positive integer m such that $\tau^{\circ m}(a_j)$ is a critical vertex. Then a_j and ∞ are joined by a new edge in $\mathcal{R}^{m+1}(S_{\mathcal{R}})$. By Lemma 6 we can isotop δ_i in $S^2 \setminus P_f$ to be disjoint from this new edge, so a_j and ∞ are in the same component of $S^2 \setminus \delta_i$. Hence $a_j \notin D_i$. So $a_j \notin D_i$ if either a_j is in a type-2 chain or we are in the situation of the left half of Figure 10.

Now suppose that a_j is in a type-1 chain that is not followed by a type-2 chain. Let a_r be the last element of the type-1 chain that contains a_j. We are in the situation of either Figure 6 or the right half of Figure 10. So there is a pair of new edges of $\mathcal{R}(S_{\mathcal{R}})$ that bounds an open disk D that contains a_r and no other postcritical points. Hence a_r cannot be in the open disk D_i since if so we can isotop D_i rel P_f into D. Similarly, in $\mathcal{R}^{r-j+1}(S_{\mathcal{R}})$ there is a pair of new edges that bounds an open disk that contains a_j and no other postcritical points.

Finally, suppose a_j is in a type-1 chain which is followed by a type-2 chain. This is the only remaining possibility. Let a_r be the last element of the type-1 chain containing a_j. Suppose that $j \neq r$. Let u be the vertex of t with label a_j. Let t' be the tile of $\mathcal{R}^{r-j}(t)$ which contains u. Then the label of u relative to t' is a_r, that is, the structure map of t' from t' to $S_{\mathcal{R}}$ maps u to a_r. Let v be the finite vertex of t' following u. The label of v relative to t' is a_{r+1}. Because (i) $j \neq r$, (ii) a_r is in a type-1 chain and (iii) a_{r+1} is in a type-2 chain, the definition of chains implies that v is not the finite vertex of t following u. So the edge e_1 of t' joining v and the ∞-vertex of t' following u must be a new edge. But, as in Figure 7, there is a new edge e_2 in $\mathcal{R}(t')$ joining v and the ∞-vertex of t' preceding u. As before, the two new edges in $\mathcal{R}^{r-j+1}(S_{\mathcal{R}})$ corresponding to e_1 and e_2 bound an open disk which contains a_j and no other postcritical point. We have reduced to the case in which $j = r$. Chains are defined so that at most one postcritical point has this property. So the open disk D_i can contain at most one postcritical point, which contradicts the assumption that δ_i is an element of a Levy cycle.

Since $\sigma_{\mathcal{R}}$ is a Thurston map whose ramification portrait is isomorphic to Γ and $\sigma_{\mathcal{R}}$ has no Levy cycle, the proof of Theorem 1 is complete. $\square$

4. Completely unobstructed portraits

Proof of Theorem 3. Suppose Γ is an abstract polynomial portrait of degree d that has at least four vertices and satisfies condition (i) or (ii) of the statement of the theorem. Suppose f is a Thurston map with portrait isomorphic to Γ, and with ∞ a fixed critical point such that $\deg_f(\infty) = d$. We prove by contradiction that f is unobstructed.

Suppose f is obstructed. Then f has a Levy cycle, and (in the terminology of [4, Section 10.3]) f has a degenerate Levy cycle $\{\delta_0, \dots, \delta_{n-1}, \delta_n = \delta_0\}$. This means the following. For each $i \in \{0, \dots, n\}$, let D_i be the disk bounded by δ_i in the 2-sphere such that $\infty \notin D_i$. For each $i \in \{1, \dots, n\}$, one component of $f^{-1}(D_i)$ is a disk D'_{i-1} such that

(a) $D_i \cap D_j = \emptyset$ if $i \neq j \in \{1, \dots, n\}$
(b) the boundary of D'_{i-1} is isotopic to δ_{i-1} rel P_f,
(c) $D'_{i-1} \cap P_f = D_{i-1} \cap P_f$, and
(d) $f|\colon D'_{i-1} \to D_i$ is a homeomorphism.

A key point for the Levy-Berstein theorem is that a postcritical point in one of the D_i's cannot be a critical point, because that would violate d). But it also cannot have an iterate that is a critical point, because that would imply that some D_j contains a critical point. Since each D_i must contain at least two postcritical points, there must be at least two postcritical points in non-attractor cycles. This gives the contradiction for case (i).

Now suppose (ii) holds. Then for some prime number p and positive integer k, we can enumerate the finite postcritical points of f as $\{v_i : 0 \leq i < p^k\}$ such that $f(v_i) = v_{i+1} \pmod{p^k}$ for every $i \in \{0, \dots, p^k - 1\}$, and if v_j is a critical value then j is a multiple of p^{k-1}. Since the sets $P_f \cap D_i$ partition the set of finite postcritical points and they all have the same cardinality, there is a positive integer m such that $\#(P_f \cap D_i) = p^m$ for all i. Then $np^m = p^k$ and $n = p^r$, where $r = k - m$. For some $i \in \{1, \dots, n\}$, $v_0 \in D_i$. Then $\{v_{jp^r}\colon 0 \leq j < p^m\} \subset D_i$ and so $\{v_{jp^{k-1}}\colon 0 \leq j < p\} \subset D_i$. Thus D_i contains every finite critical value of f.

Let D be the disk bounded by δ_i that contains ∞, and let $\widetilde{D} = f^{-1}(D)$. Then D doesn't contain any finite critical values of f. It follows that the restriction of f to $\widetilde{D} \setminus \{\infty\}$ is a covering map onto $D \setminus \{\infty\}$. But every connected covering space of a once-punctured disk is a once-punctured disk. Since f is d-to-1 near ∞, the space $\widetilde{D} \setminus \{\infty\}$ is a once-punctured disk which maps by f to $D \setminus \{\infty\}$ with degree d. Hence $\partial \widetilde{D} = f^{-1}(\partial D) = f^{-1}(\delta_i)$. This contradicts the assumption that $f^{-1}(\delta_i)$ has a connected component which maps to δ_i with degree 1. Thus f is unobstructed. $\square$

5. Rose maps

To prove Theorem 4, we need a topological description for topological polynomials which may not be subdivision maps. We begin this section by discussing our approach to this. We define a *rose* to be the boundary of the union of finitely many closed topological disks in the 2-sphere which are disjoint except for having exactly one point in common. We view a rose as a graph with exactly one vertex. Its edges are called *petals*.

Let S_1^2 and S_2^2 be two copies of S^2, and suppose that we have a finite branched covering map $g\colon S_1^2 \to S_2^2$ whose critical values lie in a finite set $P \subseteq S_2^2$. The restriction of g to $S_1^2 \setminus g^{-1}(P)$ is a covering map from $S_1^2 \setminus g^{-1}(P)$ onto $S_2^2 \setminus P$. In the context of covering maps, a straightforward thing to do in this situation is to use the fact that $S_2^2 \setminus P$ is homotopic to a rose with $\#(P)-1$ petals—the fundamental group of $S_2^2 \setminus P$ is a free group on $\#(P)-1$ generators. Let R_2 be a rose in $S_2 \setminus P$ which is a spine, and let $R_1 = g^{-1}(R_2)$. Because every connected component of $S_2^2 \setminus R_2$ contains at most one branch value of g, every connected component of $S_1^2 \setminus R_1$ is a disk, equivalently, R_1 is connected. The restriction of g to R_1 is a covering map onto R_2, and this restriction determines g up to homotopy.

With this in mind, we construct finite branched covering maps $g\colon S_1^2 \to S_2^2$ as follows. Let $R_2 \subseteq S_2^2$ be a rose with $n \geq 1$ petals. We orient S_2^2 and label n connected components of $S_2^2 \setminus R_2$ each bounded by one petal with $1,\dots,n$ in counterclockwise order. We label the remaining connected component of $S_2^2 \setminus R_2$ with ∞. Suppose that we have a finite connected graph $R_1 \subseteq S_1^2$ whose vertices have small neighborhoods which look like small neighborhoods of the vertex of R_2. Here is what this means. The connected components of $S_1^2 \setminus R_1$ are labeled with $1,\dots,n$ (duplications allowed) and ∞. We choose a barycenter for every edge of R_1 and call the resulting edges half edges. These barycenters are not vertices of R_1. Let v be a vertex of R_1. Then $2n$ half edges contain v. After orienting S_1^2, they can be written as $\epsilon_1,\dots,\epsilon_{2n}$ in counterclockwise order around v so that ϵ_{2i-1} and ϵ_{2i} are in the boundary of a connected component of $S_1 \setminus R_1$ with label i for every $i \in \{1,\dots,n\}$. Furthermore ϵ_{2i} and ϵ_{2i+1} are in the boundary of a connected component of $S_1^2 \setminus R_1$ with label ∞ for every $i \in \{1,\dots,n\}$, where $\epsilon_{2n+1} = \epsilon_1$. It is a straightforward matter, by mapping vertices, then half edges and then disks, to construct a finite branched covering map $g\colon S_1^2 \to S_2^2$ such that $g|_{R_1}$ is a covering map onto R_2 which maps vertices to vertices and edges to edges. This can be done so that g has at most one critical point in every connected component of $S_1^2 \setminus R_1$.

We define a rose map to be a map of pairs $g\colon (S_1^2, R_1) \to (S_2^2, R_2)$, where S_1^2 and S_2^2 are two oriented copies of S^2, $g\colon S_1^2 \to S_2^2$ is an orientation-preserving finite branched covering map, $R_2 \subseteq S_2^2$ is a rose, $R_1 = g^{-1}(R_2)$ is a graph with pullback graph structure and every connected component of $S_2^2 \setminus R_2$ contains at most one critical value of g. The next lemma guarantees the existence of the rose maps that we will use for the proof of Theorem

4. We will precompose a rose map $g\colon S_1^2 \to S_2^2$ with a homeomorphism $h\colon S_2^2 \to S_1^2$ to obtain a desired topological polynomial $f = g \circ h$. In the lemma, the connected components of $S_2^2 \setminus R_2$ are labeled, which induces a labeling of the connected components of $S_1^2 \setminus R_1$, which induces a labeling of the vertices of the graph dual to R_1.

Lemma 7. *Suppose that Γ is an abstract polynomial portrait whose finite postcritical vertices are $v_1, \dots, v_n$. Let u be a finite critical value of Γ with the maximum number of incoming edges from critical vertices. Let v be any finite critical value of Γ. Then there exists a rose map $g\colon (S_1^2, R_1) \to (S_2^2, R_2)$ realizing the branch data of Γ such that n connected components of $S_2^2 \setminus R_2$ each bounded by a petal of R_2 are labeled $v_1, \dots, v_n$ in counterclockwise order, the remaining connected component is labeled ∞ and R_1 has a dual graph R_1^* for which the following statements hold.*

(1) *The boundary of one connected component of $S_1^2 \setminus R_1^*$ contains exactly two critical points: one vertex with label u and one vertex with label ∞.*
(2) *If u has an incoming edge from a noncritical vertex, then the boundary of one connected component of $S_1^2 \setminus R_1^*$ contains exactly two critical points: one vertex with label v and one vertex with label ∞.*

Proof. We will construct R_1 rather explicitly.

To prepare for the construction of R_1, let U be the set of critical vertices which τ maps to u, and let W be the set of remaining finite critical vertices. So $C_\Gamma = U \amalg W \amalg \{\infty\}$. The Riemann-Hurwitz condition gives that

$$\sum_{w \in U} (\deg_\tau(w) - 1) + \sum_{w \in W} (\deg_\tau(w) - 1) = d - 1,$$

where d is the degree of Γ. So

$$\sum_{w \in W} (\deg_\tau(w) - 1) = d - 1 + \#(U) - \sum_{w \in U} \deg_\tau(w) \geq \#(U) - 1. \tag{8}$$

In particular, if W is empty, then $\#(U) = 1$. In this case there is exactly one choice for R_1 up to isomorphism. Only two connected components of $S_1^2 \setminus R_1^*$ are not monogons, and one of these has label ∞. Statement 1 is true in this case, and statement 2 is true since $u = v$. So we henceforth assume that W is nonempty.

Let m be the maximum number of incoming edges from critical vertices at the critical values other than u. Then it is possible to partition W into disjoint nonempty subsets $W_1, \dots, W_m$ so that if $w, w' \in W_i$ for some i with $w \neq w'$, then $\tau(w) \neq \tau(w')$. We do this so that if $u \neq v$, then there exists $w \in W_m$ such that $\tau(w) = v$.

Now we begin to construct R_1. We enumerate the elements of W_1, and for every $w \in W_1$ we construct a closed (2-dimensional) polygon $P_w \subseteq S_1^2$ with $\deg_\tau(w)$ sides whose interior has label $\tau(w)$. These polygons are disjoint from each other, except that each has exactly one vertex in common with the next. We obtain a chain (not to be confused with the chains in Section 3) C_1 of polygons. We also construct such chains $C_2, \ldots, C_m$ for $W_2, \ldots, W_m$ so that the chains $C_1, \ldots, C_m$ are disjoint from each other and if $u \neq v$, then the polygon with label v in C_m is last.

The choices of u and m imply that U contains at least m elements. We choose $m-1$ distinct elements $u_1, \ldots, u_{m-1} \in U$. For every $i \in \{1, \ldots, m-1\}$ we construct a polygon P_{u_i} as before which is disjoint from the polygons already constructed and from the other P_{u_j}'s, except that one vertex of P_{u_i} is a vertex of C_i that is only in the last polygon of C_i, and a different vertex of P_{u_i} is a vertex of C_{i+1} that is only in the first tile of C_{i+1}. The polygons $P_{u_1}, \ldots, P_{u_{m-1}}$ join the chains $C_1, \ldots, C_m$ to form a single chain C.

We intend to also construct similar polygons P_v for the remaining elements v of U. We intend to construct each of them in one of two ways. One way to construct P_v is to choose a vertex of C contained in only one polygon and to construct P_v so that it meets the polygons already constructed exactly in this vertex. Here is another way to construct P_v. Choose $i \in \{1, \ldots, m\}$ and two consecutive polygons P and P' in C_i. Let x be the vertex common to P and P'. We modify P and P' slightly near x, pulling them apart, so that they become disjoint. We then construct P_v so that it contains both of the new vertices in P and P' while being otherwise disjoint from the polygons already constructed.

The only obstacle to performing the constructions described in the previous paragraph is that C might not contain enough vertices to accommodate all the elements of U. But it is not difficult to see that the number of elements of U that can be accommodated, including $u_1, \ldots, u_{m-1}$, is

$$1 + \sum_{w \in W} (\deg_\tau(w) - 1).$$

Thus line 8 shows that C does indeed have enough vertices to accommodate all the elements of U. So we construct a polygon P_v as described in the previous paragraph for every $v \in U \setminus \{u_1, \ldots, u_{m-1}\}$.

Because U contains at least m elements, this can be done so that

(9) the first of these polygons meets exactly one polygon in C, the first polygon in C_1.

Furthermore, if u has an incoming edge from a noncritical vertex, then the inequality in line 8 is strict, so we may also construct these polygons so that

(10) if u has an incoming edge from a noncritical vertex, then the last polygon in C_m contains a vertex not in any of these polygons.

Every vertex in the complex constructed thus far is contained in either one or two polygons. If there are two, then their labels are different. Hence it is possible to add monogons with labeled interiors at each of these vertices so that the cyclic order of labels about each vertex agrees with the cyclic order of the labels of R_2. We have R_1. The discussion preceeding the lemma describes how to construct a rose map $g\colon (S^2_1, R_1) \to (S^2_2, R_2)$ from this information. Line 9 implies statement 1 of the lemma, and line 10 implies statement 2.

□

6. Obstructed portraits

Proof of Theorem 4. We first prove the theorem in cases (i) and (ii). Assume that Γ satisfies either (i) or (ii). We will use the following example to illustrate various constructions in the proof. Consider the abstract polynomial portrait that is shown below.

$$\begin{tikzcd}
* \arrow[dr, "3"] & & & * \arrow[d, "2"] & * \arrow[dr, "2"] & * \arrow[d, "2"] & \\
* \arrow[r, "2"] & v_0 \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[r] & v_4 \arrow[d] & \infty \arrow[loop right, "8"] \\
& & v_8 \arrow[ul] & v_7 \arrow[l] & v_6 \arrow[l] & v_5 \arrow[l] & * \arrow[l, "2"']
\end{tikzcd}$$

It satisfies the conditions of case (i).

We prepare to apply Lemma 7. Suppose that Γ has n finite postcritical vertices $v_0, \dots, v_n$ with $v_n = v_0$ such that $\tau(v_i) = v_{i+1}$ for every $i \in \{0, \dots, n-1\}$. Of course, if Γ satisfies condition (i), then $n = p^k$. Choose a finite critical value u of Γ with the maximum number of incoming edges from critical vertices. We redefine $v_0, \dots, v_n$ if necessary so that $v_0 = u$ without changing the assumptions.

In particular, if Γ satisfies (i), then there exists a critical value $v_i \in \{v_0, \dots, v_{p^k-1}\}$ such that i is not a multiple of p^{k-1}. We set $v = v_i$. In our example, we take $u = v_0$ and $v = v_2$.

To define v in the case of condition (ii), suppose that Γ satisfies (ii). Because u is in a non-attractor cycle, τ maps some vertex which is not critical to u. It follows that the inequality in line 8 is strict, and so Γ has more than one finite critical value. Let v be any finite critical value other than u.

We next define a positive integer m. If Γ satisfies (i), then we set $m = p^{k-1}$. Suppose that Γ satisfies (ii). Let i be the index such that $v = v_i$. Because n is not a prime power, it is the product of two relatively prime proper divisors. Because they are relatively prime, if both of these two proper divisors divide i, then n divides i, which is not true. Hence some positive proper divisor of n does not divide i. Let m be such a positive proper divisor of n.

So in either case (i) or (ii), m is a positive proper divisor of n such that $v = v_i$ and m does not divide i.

Lemma 7 implies that there exists a rose map $g\colon (S_1^2, R_1) \to (S_2^2, R_2)$ realizing the branch data of Γ such that n connected components of $S_2^2 \setminus R_2$ each bounded by a petal of R_2 are labeled $v_0, \dots, v_{n-1}$ in counterclockwise order, the remaining connected component is labeled ∞ and R_1 has a dual graph R_1^* for which

(1) the boundary of one connected component C_u of $S_1^2 \setminus R_1^*$ contains exactly two critical points: one vertex with label u and one vertex with label ∞;
(2) the boundary of one connected component C_v of $S_1^2 \setminus R_1^*$ contains exactly two critical points: one vertex with label v and one vertex with label ∞.

By modifying R_1^* if necessary, we may assume that the restriction of g to both C_u and C_v is injective. We identify every postcritical vertex w of Γ with a point in the connected component of $S_2^2 \setminus R_2$ with label w. These serve as the vertices of a graph R_2^* dual to R_2. Their g-pullbacks serve as the vertices of R_1^*. Figure 11 depicts important features of the rose map g for our example.

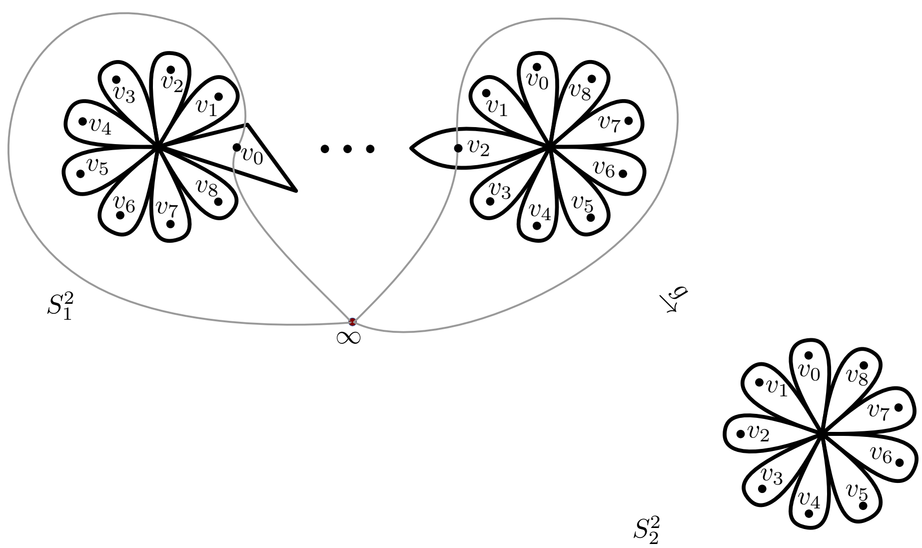


FIGURE 11. Rose map $g : S_1^2 \to S_2^2$

Now we choose m disjoint closed topological disks $D_i \subseteq S_2^2 \setminus \{\infty\}$ such that v_j is in the interior of D_i if $j \equiv i \pmod{m}$ for $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$. The restriction of g to C_v is a homeomorphism, so there exists a unique lift $\widetilde{D}_0$ of D_0 to C_v. We denote the lift of v_j to C_v by $\widetilde{v}_j$ for every index $j \equiv 0 \pmod{m}$. There likewise exist unique lifts $\widetilde{D}_i$ of D_i to C_u for every $i \in \{1, \ldots, m-1\}$. We denote the lift of v_j to C_u by $\widetilde{v}_j$ for every index $j \not\equiv 0 \pmod{m}$. Figure 12 shows all of these points and disks for our example.

We next construct an orientation-preserving homeomorphism $h\colon S_2^2 \to S_1^2$ such that (i) $h(v_j) = \widetilde{v}_{j+1}$ for $j \in \{0, \ldots, n-1\}$, (ii) $h(\infty) = g^{-1}(\infty)$ and (iii) $h(D_i) = \widetilde{D}_{i+1}$ for $i \in \{0, \ldots, m-1\}$, where $D_m = D_0$. See Figure 13.

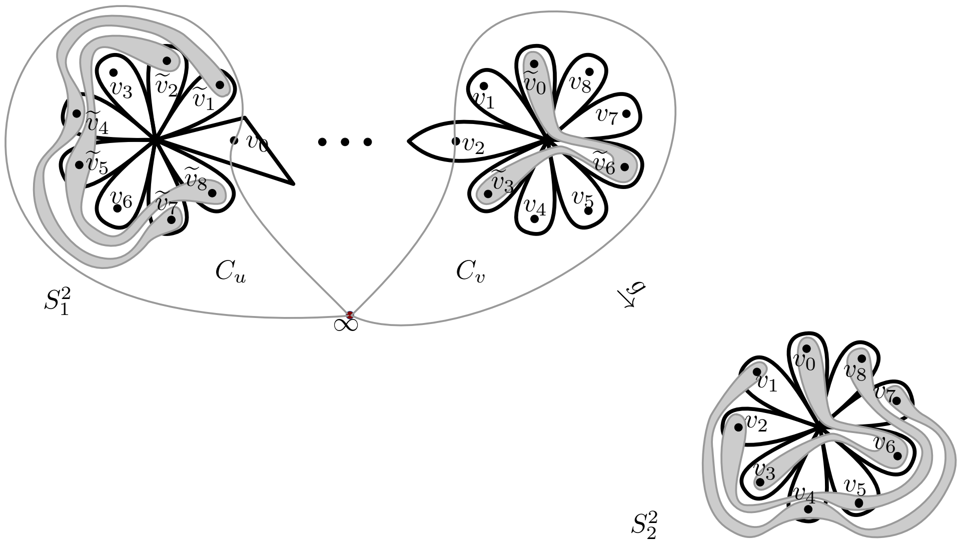


FIGURE 12. Rose map $g : S_1^2 \to S_2^2$

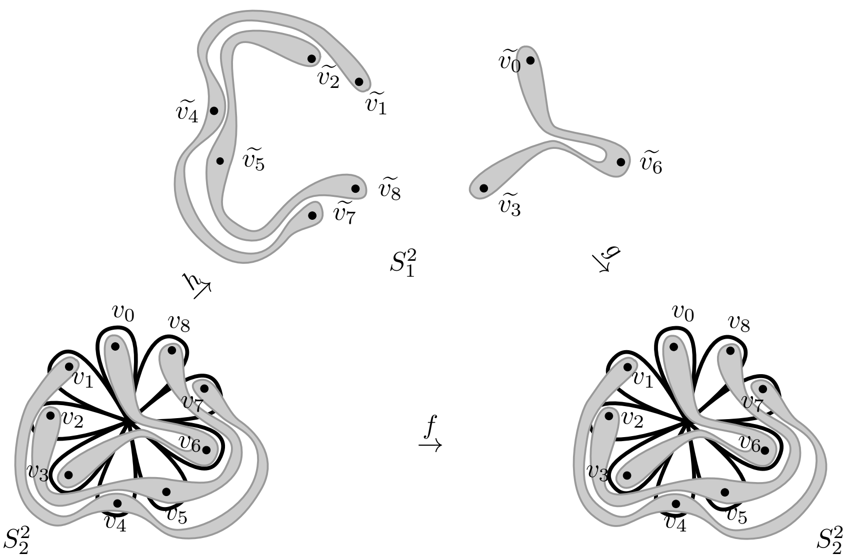


FIGURE 13. The obstructed Thurston map $f : S_2^2 \to S_2^2$

Finally, define $f := g \circ h$. The map f is a Thurston map whose portrait is isomorphic to Γ and the boundaries of the disks D_i form a degenerate Levy cycle. This proves Theorem 4 in cases (i) and (ii).

Now suppose that Γ satisfies (iii). Let C be a non-attractor cycle of length $\ell \geq 2$ that does not contain all of the finite critical values of Γ. Let $v_1, \ldots, v_n$ be the finite postcritical vertices of Γ indexed so that $v_1, \ldots, v_\ell$ are the vertices of C, $\tau(v_\ell) = v_1$ and $\tau(v_i) = v_{i+1}$ for $i \in \{1, \ldots, \ell - 1\}$. By Lemma

7 there exists a critical value $v \notin C$ and a rose map $g\colon (S^2_1, R_1) \to (S^2_2, R_2)$ realizing the branch data of Γ such that n connected components of $S^2_2 \setminus R_2$ each bounded by a petal of R_2 are labeled $v_1, \dots, v_n$ in counterclockwise order, the remaining connected component is labeled ∞ and the boundary of one connected component C_v of $S_1 \setminus R^*_1$ contains exactly two critical points: one vertex with label v and one vertex with label ∞. We may assume that the restriction of g to C_v is injective. As in cases (i) and (ii), we identify every postcritical vertex w of Γ with a point in the connected component of $S^2_2 \setminus R_2$ with label w.

Let D be a closed topological disk in S^2_2 whose interior contains $v_1, \dots, v_\ell$ but such that D contains no other postcritical vertex of Γ. Let $\widetilde{D}$ be the lift of D to C_v. Let $\widetilde{v}_i$ be the lift of v_i to C_v for $i \in \{1, \dots, \ell\}$. Now construct an orientation-preserving homeomorphism $h\colon S^2_2 \to S^2_1$ such that (i) $h(v_i) = \widetilde{v}_{i+1}$ for $i \in \{1, \dots, \ell - 1\}$, (ii) $h(v_\ell) = v_1$, (iii) $h(\infty) = g^{-1}(\infty)$, (iv) $h(D) = \widetilde{D}$ and (v) the portrait of $f := g \circ h$ is isomorphic to Γ. Since ∂D is by itself a degenerate Levy cycle for f, this proves Theorem 4 in case (iii).

Finally, we consider case (iv), where Γ has at least two non-attractor cycles of length one. Let a and b denote the vertices of two non-attractor cycles of Γ of length 1. Define the abstract portrait Γ' as follows: $V(\Gamma') = V(\Gamma)$, $\tau'(a) = b$, $\tau'(b) = a$, and $\tau'(x) = \tau(x)$ for all $x \in V(\Gamma') \setminus \{a, b\}$. So Γ' satisfies (iii). The proof for case (iii) shows that there is a Thurston map f realizing Γ' that has a degenerate Levy cycle consisting of a single curve which bounds a disk D such that $D \cap P_f = \{a, b\}$. Postcompose f with an orientation-preserving homeomorphism $h : S^2 \to S^2$ such that h is the identity in the complement of D, $h(b) = a$ and $h(a) = b$. Then $h \circ f$ is an obstructed Thurston map whose portrait is isomorphic to Γ. □

7. Classification of completely unobstructed portraits

Proof of Theorem 5. For the forward direction, we will prove the contrapositive. Suppose that Γ is an abstract polynomial portrait that does not satisfy any of the conditions (i)-(iv). Since Γ doesn't satisfy (i) and (ii), Γ has at least four postcritical vertices and there is a non-attractor cycle. Since Γ doesn't satisfy (iii), either Γ has at least two non-attractor cycles or there is a non-attractor cycle of length at least two. If Γ has at least two nonattractor cycles, then it satisfies conditions (iii) or (iv) of Theorem 4 and so Γ can be realized by an obstructed Thurston map. Now suppose that Γ has a single non-attractor cycle and it has length at least two. If this cycle doesn't contain all of the finite postcritical vertices, then Γ can be realized by an obstructed map by condition (iii) of Theorem 4. So we may suppose that all of the finite postcritical vertices are in a single non-attractor cycle. If its length is not a prime power, then Γ can be realized by an obstructed map by condtion (ii) of Theorem 4. If its length is a prime power, then since

Γ doesn't satisfy (iv) it satisfies (i) of Theorem 4 and Γ can be realized by an obstructed map.

For the reverse direction, suppose Γ is an abstract polynomial portrait and f is a Thurston map whose portrait is isomorphic to Γ. If Γ satisfies (i), then f is unobstructed by Thurston's characterization theorem since there aren't enough postcritical points to have an obstruction. If Γ satifies (ii), then f is unobstructed by the Levy-Berstein theorem. If Γ satisfies (iii) or (iv), then f is unobstructed by Theorem 3. $\square$

References

[1] Krzysztof Barański. On realizability of branched coverings of the sphere. *Topology Appl.*, 116(3):279–291, 2001.

[2] Adrien Douady and John H. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.*, 171(2):263–297, 1993.

[3] Allan L. Edmonds, Ravi S. Kulkarni, and Robert E. Stong. Realizability of branched coverings of surfaces. *Trans. Amer. Math. Soc.*, 282(2):773–790, 1984.

[4] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2*. Matrix Editions, Ithaca, NY, 2016. Surface homeomorphisms and rational functions.

[5] Gregory A. Kelsey. Mapping schemes realizable by obstructed topological polynomials. *Conform. Geom. Dyn.*, 16:44–80, 2012.

[6] A. G. Khovanskii and Smilka Zdravkovska. Branched covers of S^2 and braid groups. *J. Knot Theory Ramifications*, 5(1):55–75, 1996.

[7] Silvio Vieira Ferreira Levy. *CRITICALLY FINITE RATIONAL MAPS (THURSTON)*. ProQuest LLC, Ann Arbor, MI, 1985. Thesis (Ph.D.)–Princeton University.

[8] René Thom. L'équivalence d'une fonction différentiable et d'un polynome. *Topology*, 3(suppl, suppl. 2):297–307, 1965.

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