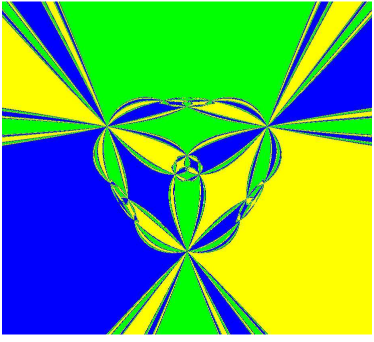
Rational maps of $\mathbb{CP}^2$ with equal dynamical degrees, no invariant foliation, and two *different* measures of maximal entropy

Jeffrey Diller[1], Scott Kaschner[2], Rodrigo Pérez[3], and Roland Roeder[3]

Notre Dame[1], Butler[2] and IUPUI[3]

Bremen, August 2015



Plan for the talk

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example
   (b) General Criteria

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example
   (b) General Criteria
3. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees, no invariant foliation, and two different measures of maximal entropy. (With Diller).

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example
   (b) General Criteria
3. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees, no invariant foliation, and two different measures of maximal entropy. (With Diller).
   (a) The perturbed Chebyshev map (a specific example).

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example
   (b) General Criteria
3. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees, no invariant foliation, and two different measures of maximal entropy. (With Diller).
   (a) The perturbed Chebyshev map (a specific example).
   (b) General construction.

Plan for the talk

1. Dynamical Degrees and Entropy Conjecture for Rational Maps
   (a) Guedj's Question about maps with equal maximal dynamical degrees.
2. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees and no invariant foliation. (With Kaschner and Pérez)
   (a) Specific example
   (b) General Criteria
3. Rational Maps of $\mathbb{CP}^2$ with equal dynamical degrees, no invariant foliation, and two different measures of maximal entropy. (With Diller).
   (a) The perturbed Chebyshev map (a specific example).
   (b) General construction.

Goal: convince you that it's interesting to study rational maps of $\mathbb{CP}^2$ having equal dynamical degrees.

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$.

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$.
Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$.
Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n\to\infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$. Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \left(1 \leq p \leq \dim(M)\right).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n \to \infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Favorite example: $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a rational map.

$$f[X:Y:Z] = [f_1(X,Y,Z), f_2(X,Y,Z), f_3(X,Y,Z)],$$

where f_1, f_2, and f_3 are homogeneous polynomials all of the same degree having no common factor of positive degree.

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$. Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n \to \infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Favorite example: $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a rational map.

$$f[X : Y : Z] = [f_1(X,Y,Z), f_2(X,Y,Z), f_3(X,Y,Z)],$$

where f_1, f_2, and f_3 are homogeneous polynomials all of the same degree having no common factor of positive degree. (Denoted $\deg_{\mathrm{alg}}(f)$.)

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$. Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n \to \infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Favorite example: $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a rational map.

$$f[X : Y : Z] = [f_1(X, Y, Z), f_2(X, Y, Z), f_3(X, Y, Z)],$$

where f_1, f_2, and f_3 are homogeneous polynomials all of the same degree having no common factor of positive degree. (Denoted $\deg_{\mathrm{alg}}(f)$.)

$$\mathcal{I}_f = \{[X : Y : Z] \in \mathbb{CP}^2 \; : \; f_1 = f_2 = f_3 = 0\}$$

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$. Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n \to \infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Favorite example: $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a rational map.

$$f[X : Y : Z] = [f_1(X,Y,Z), f_2(X,Y,Z), f_3(X,Y,Z)],$$

where f_1, f_2, and f_3 are homogeneous polynomials all of the same degree having no common factor of positive degree. (Denoted $\deg_{\mathrm{alg}}(f)$.)

$$\mathcal{I}_f = \{[X : Y : Z] \in \mathbb{CP}^2 \ : \ f_1 = f_2 = f_3 = 0\}$$

- $\lambda_1(f) = \lim \left(\deg_{\mathrm{alg}}(f^n)\right)^{1/n}$.

Dynamical Degrees

Let $f : M \dashrightarrow M$ be a meromorphic self-map of a compact Kähler manifold.

Such map is not continuous on a codimension ≥ 2 indeterminacy set $\mathcal{I}_f$. Nevertheless, f induces a well-defined pullback action

$$f^* : H^{p,p}(M) \to H^{p,p}(M) \quad \big(1 \leq p \leq \dim(M)\big).$$

For $1 \leq p \leq \dim(M)$, the p-th dynamical degree is

$$\lambda_p(f) = \lim_{n\to\infty} \left\| \left(f^n\right)^* : H^{p,p}(M) \to H^{p,p}(M) \right\|^{1/n}.$$

Favorite example: $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a rational map.

$$f[X : Y : Z] = [f_1(X, Y, Z), f_2(X, Y, Z), f_3(X, Y, Z)],$$

where f_1, f_2, and f_3 are homogeneous polynomials all of the same degree having no common factor of positive degree. (Denoted $\deg_{\mathrm{alg}}(f)$.)

$$\mathcal{I}_f = \{[X : Y : Z] \in \mathbb{CP}^2 \ : \ f_1 = f_2 = f_3 = 0\}$$

- $\lambda_1(f) = \lim \left(\deg_{\mathrm{alg}}(f^n)\right)^{1/n}$.
- $\lambda_2(f) = \deg_{\mathrm{top}}(f)$ is the topological degree of f.

Entropy Conjecture for Meromorphic Maps*

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

---

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyapunov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

---

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

Builds from early work of Bedford-Lyubich-Smillie, Briend-Duval, Hubbard-Papadapol, Hubbard-Oberst-Vorth, Fornaess-Sibony, Ueda, ...

---

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

Builds from early work of Bedford-Lyubich-Smillie, Briend-Duval, Hubbard-Papadapol, Hubbard-Oberst-Vorth, Fornaess-Sibony, Ueda, ...

Proved cases:

- $\lambda_{\dim(X)}(f)$ is maximal (Guedj, Dinh-Nguyen-Truong).

---

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

Builds from early work of Bedford-Lyubich-Smillie, Briend-Duval, Hubbard-Papadapol, Hubbard-Oberst-Vorth, Fornaess-Sibony, Ueda, ...

Proved cases:

- $\lambda_{\dim(X)}(f)$ is maximal (Guedj, Dinh-Nguyen-Truong).
- $\dim(X) = 2$, $\lambda_1(f) > \lambda_2(f)$ (Diller-Dujardin-Guedj).

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Entropy Conjecture for Meromorphic Maps*

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

Builds from early work of Bedford-Lyubich-Smillie, Briend-Duval, Hubbard-Papadapol, Hubbard-Oberst-Vorth, Fornaess-Sibony, Ueda, ...

Proved cases:

- $\lambda_{\dim(X)}(f)$ is maximal (Guedj, Dinh-Nguyen-Truong).
- $\dim(X) = 2$, $\lambda_1(f) > \lambda_2(f)$ (Diller-Dujardin-Guedj).
- Generic rational maps of $\mathbb{CP}^k$. (Vigny).

*Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.
Prototypical example: Endomorphisms $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

$$\lambda_1(f) = \deg_{\mathrm{alg}}(f) < \left(\deg_{\mathrm{alg}}(f)\right)^2 = \lambda_2(f).$$

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.
Prototypical example: Endomorphisms $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

$$\lambda_1(f) = \deg_{\mathrm{alg}}(f) < \left(\deg_{\mathrm{alg}}(f)\right)^2 = \lambda_2(f).$$

In general, there exists a "repelling" measure of maximal entropy.
(Unique.)

Case 3 Equal degrees: $\lambda_1(f) = \lambda_2(f)$.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy. (Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.
Prototypical example: Endomorphisms $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

$$\lambda_1(f) = \deg_{\mathrm{alg}}(f) < \left(\deg_{\mathrm{alg}}(f)\right)^2 = \lambda_2(f).$$

In general, there exists a "repelling" measure of maximal entropy. (Unique.)

Case 3 Equal degrees: $\lambda_1(f) = \lambda_2(f)$.
No prototypical example.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.
Prototypical example: Endomorphisms $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

$$\lambda_1(f) = \deg_{\mathrm{alg}}(f) < \left(\deg_{\mathrm{alg}}(f)\right)^2 = \lambda_2(f).$$

In general, there exists a "repelling" measure of maximal entropy.
(Unique.)

Case 3 Equal degrees: $\lambda_1(f) = \lambda_2(f)$.
No prototypical example.
Little known.

Special case of rational maps $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

Case 1 Small topological degree: $\lambda_1(f) > \lambda_2(f)$.
Prototypical example: Hénon maps $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$.

$$\lambda_1(h) = \deg_{\mathrm{alg}}(h) > 1 = \lambda_2(h).$$

In general, there exists a "saddle" measure of maximal entropy.
(Known to be unique in several cases. Believed in all.)

Case 2 Large topological degree: $\lambda_1(f) < \lambda_2(f)$.
Prototypical example: Endomorphisms $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.

$$\lambda_1(f) = \deg_{\mathrm{alg}}(f) < \left(\deg_{\mathrm{alg}}(f)\right)^2 = \lambda_2(f).$$

In general, there exists a "repelling" measure of maximal entropy.
(Unique.)

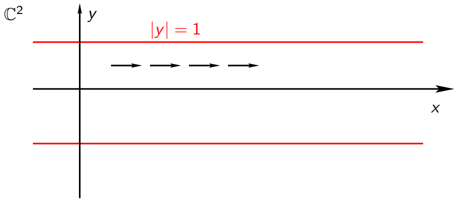
Case 3 Equal degrees: $\lambda_1(f) = \lambda_2(f)$.
No prototypical example.
Little known.
Subject of today's talk.

What can go wrong when map is not cohomologically hyperbolic?

Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

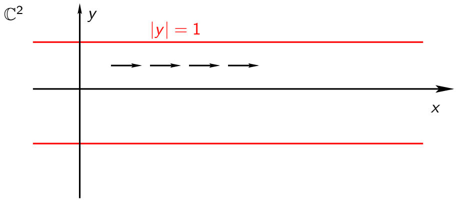
What can go wrong when map is not cohomologically hyperbolic?

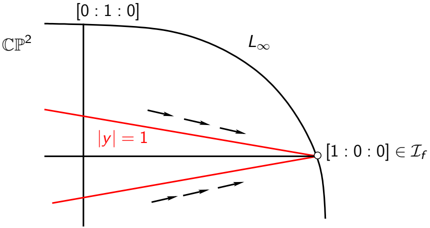
Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x, y) = (x + 1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f)$.



What can go wrong when map is not cohomologically hyperbolic?

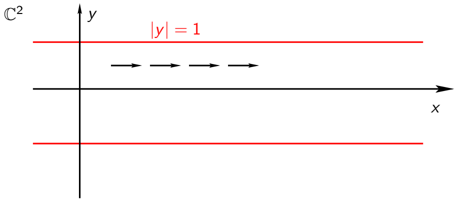
Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

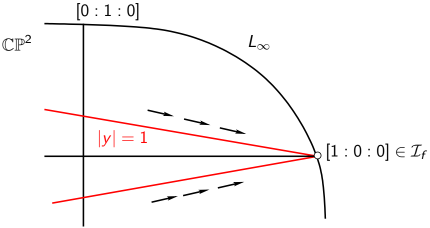




What can go wrong when map is not cohomologically hyperbolic?

Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

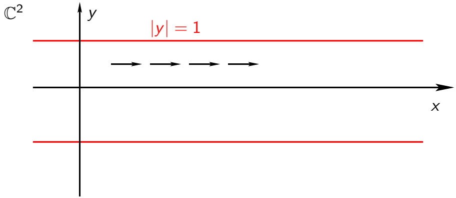


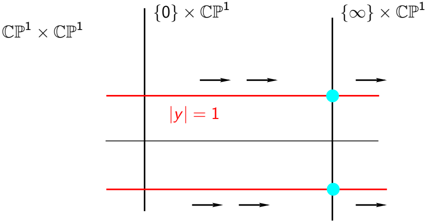


Can't have a measure of entropy $\log 2 = \log \lambda_1(f) = \log \lambda_2(f)$.

What can go wrong when map is not cohomologically hyperbolic?

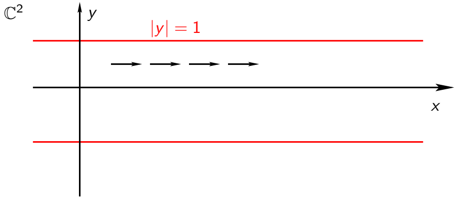
Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

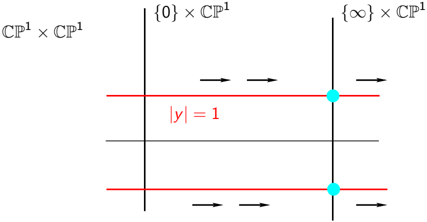




What can go wrong when map is not cohomologically hyperbolic?

Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

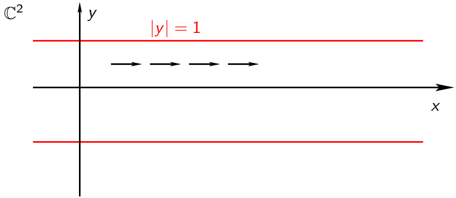


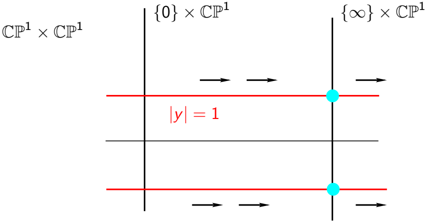


Lebesgue measure on $\{\infty\} \times \{|y| = 1\}$ has entropy $\log 2$.

What can go wrong when map is not cohomologically hyperbolic?

Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f).$

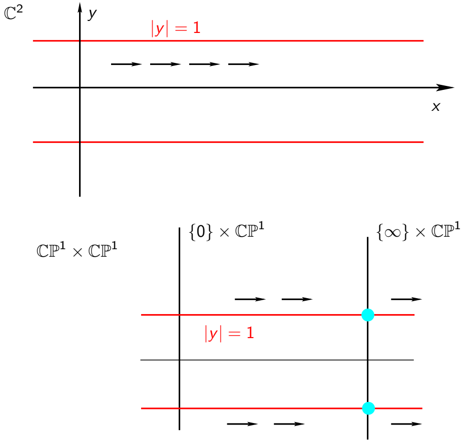




Lebesgue measure on $\{\infty\} \times \{|y| = 1\}$ has entropy $\log 2$.
However, it's supported on an algebraic curve $\infty \times \mathbb{CP}^1$.

What can go wrong when map is not cohomologically hyperbolic?

Guedj example: $f : \mathbb{C}^2 \to \mathbb{C}^2, \quad f(x,y) = (x+1, y^2), \quad \lambda_1(f) = 2 = \lambda_2(f)$.



Lebesgue measure on $\{\infty\} \times \{|y| = 1\}$ has entropy $\log 2$.
However, it's supported on an algebraic curve $\infty \times \mathbb{CP}^1$.
Can disappear under birational conjugacy.

Entropy Conjecture for Meromorphic Maps†

Definition The map f is cohomologically hyperbolic if one of its dynamical degrees is larger than all the others.

Conjecture Let $f : X \dashrightarrow X$ be a cohomologically hyperbolic rational map with maximal dynamical degree λ_ℓ.

Then there is a unique invariant probability measure μ of maximal entropy that does not charge hypersurfaces with the following properties:

(i) μ is mixing and has entropy $h_\mu(f) = \log \lambda_\ell(f)$.

(ii) μ is hyperbolic and the Lyaponov exponents are bounded away from 0 by $\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell-1}(f)$ and $-\frac{1}{2} \log \lambda_\ell(f)/\lambda_{\ell+1}(f)$.

(iii) Periodic points with ℓ repelling directions and $\dim(X) - \ell$ attracting directions are asymptotically equidistributed with respect to μ.

† Vincent Guedj. *ETDS*, 25(6):1847–1855, 2005.

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X.

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X. The fibers of π give a singular foliation of X.

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X. The fibers of π give a singular foliation of X.

φ preserves the fibration defined by π if there is an induced map $\xi : Y \dashrightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \overset{\xi}{\dashrightarrow} & Y. \end{array}$$

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X. The fibers of π give a singular foliation of X.

φ preserves the fibration defined by π if there is an induced map $\xi : Y \dashrightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \overset{\xi}{\dashrightarrow} & Y. \end{array}$$

Fancy way of saying that f is a skew product.

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X. The fibers of π give a singular foliation of X.

φ preserves the fibration defined by π if there is an induced map $\xi : Y \dashrightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \overset{\xi}{\dashrightarrow} & Y. \end{array}$$

Fancy way of saying that f is a skew product.

- Surface bimeromorphisms that are not cohomologically hyperbolic (i.e., $\lambda_1(\varphi) = \lambda_2(\varphi) = 1$) always preserve an invariant fibration (Diller-Favre 2001).

How else can non cohomologically hyperbolic maps degenerate?

When $\dim Y < \dim X$, any dominant meromorphic map $\pi : X \dashrightarrow Y$ with connected fibers induces a fibration on X. The fibers of π give a singular foliation of X.

φ preserves the fibration defined by π if there is an induced map $\xi : Y \dashrightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Y & \overset{\xi}{\dashrightarrow} & Y. \end{array}$$

Fancy way of saying that f is a skew product.

- Surface bimeromorphisms that are not cohomologically hyperbolic (i.e., $\lambda_1(\varphi) = \lambda_2(\varphi) = 1$) always preserve an invariant fibration (Diller-Favre 2001).
- Meromorphic maps that are not cohomologically hyperbolic arise naturally when studying the spectral theory of operators on self-similar spaces. All the examples studied in that context preserve an invariant fibration (Sabot, Bartholdi, Grigorchuk, Zuk early 2000s).

Guedj's Question

Based on the previous examples, Guedj (2006) asked whether every non-cohomologically hyperbolic map preserves a fibration.

Guedj's Question

Based on the previous examples, Guedj (2006) asked whether every non-cohomologically hyperbolic map preserves a fibration.

(Bedford–Cantat–Kim) give a family of examples of pseudo-automorphisms of a blow-up of $\mathbb{CP}^3$ with equal 1^{st} and 2^{nd} dynamical degrees for which no iterate preserves a singular holomorphic foliation of dimension 1 or 2.

Guedj's Question

Based on the previous examples, Guedj (2006) asked whether every non-cohomologically hyperbolic map preserves a fibration.

(Bedford–Cantat–Kim) give a family of examples of pseudo-automorphisms of a blow-up of $\mathbb{CP}^3$ with equal 1^{st} and 2^{nd} dynamical degrees for which no iterate preserves a singular holomorphic foliation of dimension 1 or 2.

Theorem 1 (Kaschner–Pérez-R) The rational map $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ given by

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X - Z) : -(X + Z)(X - Z)]$$

is not cohomologically hyperbolic ($\lambda_1(\varphi) = \lambda_2(\varphi) = 2$), and no iterate of φ preserves a singular holomorphic foliation.

Guedj's Question

Based on the previous examples, Guedj (2006) asked whether every non-cohomologically hyperbolic map preserves a fibration.

(Bedford–Cantat–Kim) give a family of examples of pseudo-automorphisms of a blow-up of $\mathbb{CP}^3$ with equal 1^{st} and 2^{nd} dynamical degrees for which no iterate preserves a singular holomorphic foliation of dimension 1 or 2.

Theorem 1 (Kaschner–Pérez-R) The rational map $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ given by

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X - Z) : -(X + Z)(X - Z)]$$

is not cohomologically hyperbolic ($\lambda_1(\varphi) = \lambda_2(\varphi) = 2$), and no iterate of φ preserves a singular holomorphic foliation.

Preserving a fibration is stronger than preserving a foliation, so the answer to Guedj's question is "no".

Singular Holomorphic Foliations on Complex Surfaces I

Singular Holomorphic Foliations on Complex Surfaces I

The standard holomorphic foliation on $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

The standard holomorphic foliation on $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

A holomorphic foliation $\mathcal{F}$ on a (possibly non-compact) complex surface X is the partition

$$X = \bigsqcup_{\alpha} L_\alpha$$

of X into a disjoint union of connected subsets L_α, which is locally biholomorphically equivalent to the standard holomorphic foliation on $\mathbb{D}^2$.

Singular Holomorphic Foliations on Complex Surfaces I

The standard holomorphic foliation on $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

A holomorphic foliation $\mathcal{F}$ on a (possibly non-compact) complex surface X is the partition

$$X = \bigsqcup_{\alpha} L_\alpha$$

of X into a disjoint union of connected subsets L_α, which is locally biholomorphically equivalent to the standard holomorphic foliation on $\mathbb{D}^2$.

Each L_α is a Riemann surface called a leaf of $\mathcal{F}$.

Singular Holomorphic Foliations on Complex Surfaces I

The standard holomorphic foliation on $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

A holomorphic foliation $\mathcal{F}$ on a (possibly non-compact) complex surface X is the partition

$$X = \bigsqcup_{\alpha} L_\alpha$$

of X into a disjoint union of connected subsets L_α, which is locally biholomorphically equivalent to the standard holomorphic foliation on $\mathbb{D}^2$.

Each L_α is a Riemann surface called a leaf of $\mathcal{F}$.

A singular holomorphic foliation on X is a foliation $\mathcal{F}$ on $X \setminus P$, where P is a discrete set of points through which $\mathcal{F}$ does not extend.

Singular Holomorphic Foliations on Complex Surfaces I

The standard holomorphic foliation on $\mathbb{D}^2$ is the representation of the bidisk $\mathbb{D}^2$ as the disjoint union of complex one-dimensional disks

$$\mathbb{D}^2 = \bigsqcup_{|y|<1} \{|x| < 1\} \times \{y\}.$$

A holomorphic foliation $\mathcal{F}$ on a (possibly non-compact) complex surface X is the partition

$$X = \bigsqcup_{\alpha} L_\alpha$$

of X into a disjoint union of connected subsets L_α, which is locally biholomorphically equivalent to the standard holomorphic foliation on $\mathbb{D}^2$.

Each L_α is a Riemann surface called a leaf of $\mathcal{F}$.

A singular holomorphic foliation on X is a foliation $\mathcal{F}$ on $X \setminus P$, where P is a discrete set of points through which $\mathcal{F}$ does not extend.

The points of P are called the singular points of $\mathcal{F}$.

Singular Holomorphic Foliations on Complex Surfaces II

Singular Holomorphic Foliations on Complex Surfaces II

Theorem (Ilyashenko '72) In a neighborhood of each singular point $p \in X$, $\mathcal{F}$ is generated as the integral curves of some holomorphic 1-form.

Singular Holomorphic Foliations on Complex Surfaces II

Theorem (Ilyashenko '72) In a neighborhood of each singular point $p \in X$, $\mathcal{F}$ is generated as the integral curves of some holomorphic 1-form.

This allows us to express $\mathcal{F}$ by an open cover $\{U_i\}$ of X and a system of holomorphic one-forms ω_i on U_i with isolated zeros, satisfying the compatibility condition that

$$\omega_i = g_{ij}\omega_j$$

for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

Singular Holomorphic Foliations on Complex Surfaces II

Theorem (Ilyashenko '72) In a neighborhood of each singular point $p \in X$, $\mathcal{F}$ is generated as the integral curves of some holomorphic 1-form.

This allows us to express $\mathcal{F}$ by an open cover $\{U_i\}$ of X and a system of holomorphic one-forms ω_i on U_i with isolated zeros, satisfying the compatibility condition that

$$\omega_i = g_{ij}\omega_j$$

for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

The zeros of ω_i correspond to the singular points of $\mathcal{F}$.

Theorem (Ilyashenko '72) In a neighborhood of each singular point $p \in X$, $\mathcal{F}$ is generated as the integral curves of some holomorphic 1-form.

This allows us to express $\mathcal{F}$ by an open cover $\{U_i\}$ of X and a system of holomorphic one-forms ω_i on U_i with isolated zeros, satisfying the compatibility condition that

$$\omega_i = g_{ij}\omega_j$$

for some non-vanishing holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}$.

The zeros of ω_i correspond to the singular points of $\mathcal{F}$.

Lemma 1 Let $\mathcal{F}$ be a foliation on a surface X and suppose $C \subset X$ is an irreducible algebraic curve. Then, either C is a leaf of $\mathcal{F}$ or C is transverse to $\mathcal{F}$ away from finitely many points.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Lemma 2 Let $\varphi : X \to Y$ be finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Lemma 2 Let $\varphi : X \to Y$ be finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.

From now on $\varphi : X \dashrightarrow Y$ will stand for a dominant rational map between complex projective algebraic surfaces.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Lemma 2 Let $\varphi : X \to Y$ be finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.

From now on $\varphi : X \dashrightarrow Y$ will stand for a dominant rational map between complex projective algebraic surfaces.

Let $\mathcal{I}_\varphi$ denote the indeterminacy locus of φ. (Finite set.)

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Lemma 2 Let $\varphi : X \to Y$ be finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.

From now on $\varphi : X \dashrightarrow Y$ will stand for a dominant rational map between complex projective algebraic surfaces.

Let $\mathcal{I}_\varphi$ denote the indeterminacy locus of φ. (Finite set.)

Lemma 3 $\left(\varphi|_{X \setminus \mathcal{I}_\varphi}\right)^* \mathcal{F}$ has finitely many singular points on $X \setminus \mathcal{I}_\varphi$.

Pullback of foliations

Let $\varphi : X \to Y$ be a dominant holomorphic map between two surfaces and let $\mathcal{F}$ be a foliation on Y given by $\{(U_i, \omega_i)\}$.

The pullback $\varphi^*\mathcal{F}$ is defined by $\{(\varphi^{-1}(U_i), \varphi^{\#}\omega_i)\}$, where each form $\varphi^{\#}\omega_i$ is obtained by rescaling $\varphi^*\omega_i$, i.e., dividing it by a suitable holomorphic function, in order to eliminate any non-isolated zeros.

Lemma 2 Let $\varphi : X \to Y$ be finite at $x \in X$ and let $\mathcal{F}$ be a foliation on Y. If $\varphi(x)$ is a singular point for $\mathcal{F}$ then x is a singular point for $\varphi^*(\mathcal{F})$.

From now on $\varphi : X \dashrightarrow Y$ will stand for a dominant rational map between complex projective algebraic surfaces.

Let $\mathcal{I}_\varphi$ denote the indeterminacy locus of φ. (Finite set.)

Lemma 3 $\left(\varphi|_{X \setminus \mathcal{I}_\varphi}\right)^* \mathcal{F}$ has finitely many singular points on $X \setminus \mathcal{I}_\varphi$.

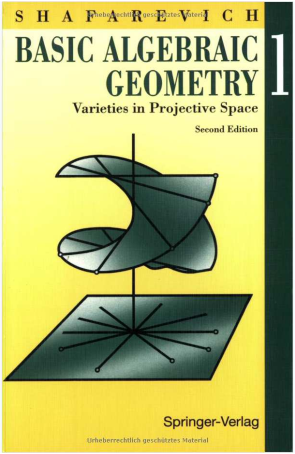
Thus, we can define $\varphi^*\mathcal{F}$ to be the extension of $\left(\varphi|_{X \setminus \mathcal{I}_\varphi}\right)^* \mathcal{F}$ through $\mathcal{I}_\varphi$.

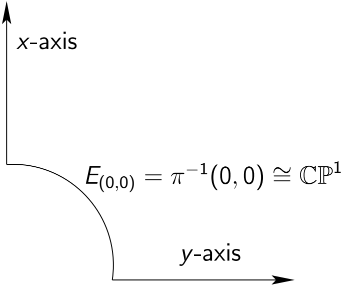
Blow up of $\mathbb{C}^2$ at $(0,0)$

$$\widetilde{\mathbb{C}^2} := \{((x,y),l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid (x,y) \in l\}.$$

Blow up of $\mathbb{C}^2$ at $(0, 0)$

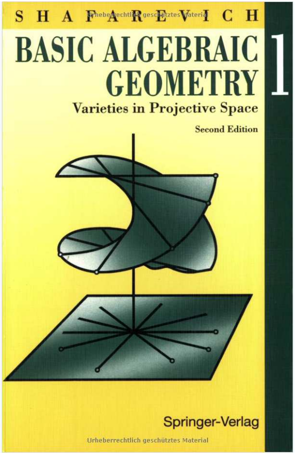
$$\widetilde{\mathbb{C}^2} := \{((x, y), l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid (x, y) \in l\}.$$

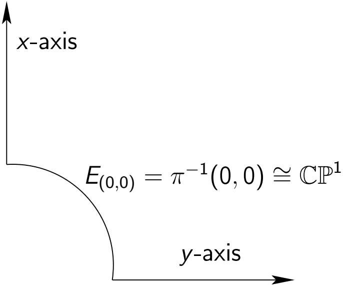




Blow up of $\mathbb{C}^2$ at $(0,0)$

$$\widetilde{\mathbb{C}^2} := \{((x,y),l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid (x,y) \in l\}.$$

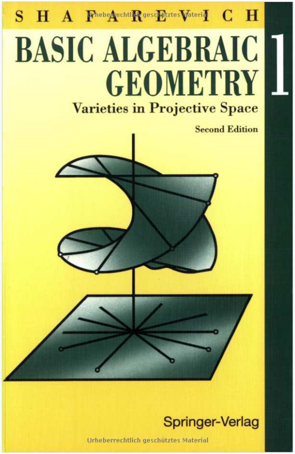


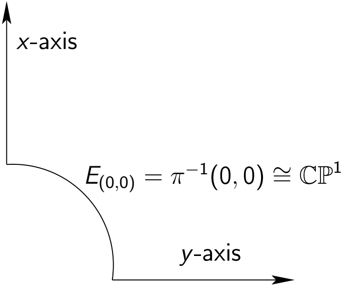


Canonical Projection: $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$, $\pi(((x,y),l)) = (x,y)$.

Blow up of $\mathbb{C}^2$ at $(0,0)$

$$\widetilde{\mathbb{C}^2} := \{((x,y),l) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid (x,y) \in l\}.$$





Canonical Projection: $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$, $\pi(((x,y),l)) = (x,y)$.

Exceptional Divisor: $E_{(0,0)} := \pi^{-1}(0,0) \cong \mathbb{CP}^1$.

Blow-ups II

Blow-ups II

Classical Theorem: There is a sequence of blow-ups $\pi : \widetilde{X} \to X$ over $\mathcal{I}_\varphi$ so that φ lifts to a holomorphic map $\widetilde{\varphi}$ making the following diagram commute

$$\begin{array}{ccc} \widetilde{X} & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\widetilde{\varphi}} & \\ X & \overset{\varphi}{\dashrightarrow} & Y, \end{array}$$

wherever $\widetilde{\varphi}$ and $\varphi \circ \pi$ are both defined.

Blow-ups II

Classical Theorem: There is a sequence of blow-ups $\pi : \widetilde{X} \to X$ over $\mathcal{I}_\varphi$ so that φ lifts to a holomorphic map $\widetilde{\varphi}$ making the following diagram commute

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\widetilde{\varphi}} & \\ X & \overset{\varphi}{\dashrightarrow} & Y, \end{array}$$

wherever $\widetilde{\varphi}$ and $\varphi \circ \pi$ are both defined.

We say that the pair $\pi, \widetilde{\varphi}$ resolves the indeterminacy of φ.

Blow-ups II

Classical Theorem: There is a sequence of blow-ups $\pi : \widetilde{X} \to X$ over $\mathcal{I}_\varphi$ so that φ lifts to a holomorphic map $\widetilde{\varphi}$ making the following diagram commute

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\widetilde{\varphi}} & \\ X & \overset{\varphi}{\dashrightarrow} & Y, \end{array}$$

wherever $\widetilde{\varphi}$ and $\varphi \circ \pi$ are both defined.

We say that the pair $\pi, \widetilde{\varphi}$ resolves the indeterminacy of φ.

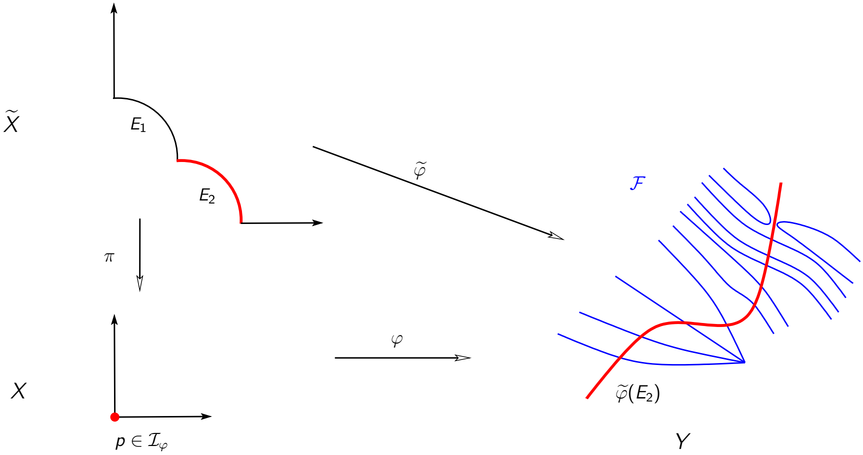
See, for example, Shafarevich.

Generation of singularities

Lemma 4 Let $\varphi : X \dashrightarrow Y$ be a dominant rational map of surfaces, let $p \in \mathcal{I}_\varphi$, and suppose $\pi : \widetilde{X} \to X$ and $\widetilde{\varphi} : \widetilde{X} \to Y$ resolve the indeterminacy of φ.

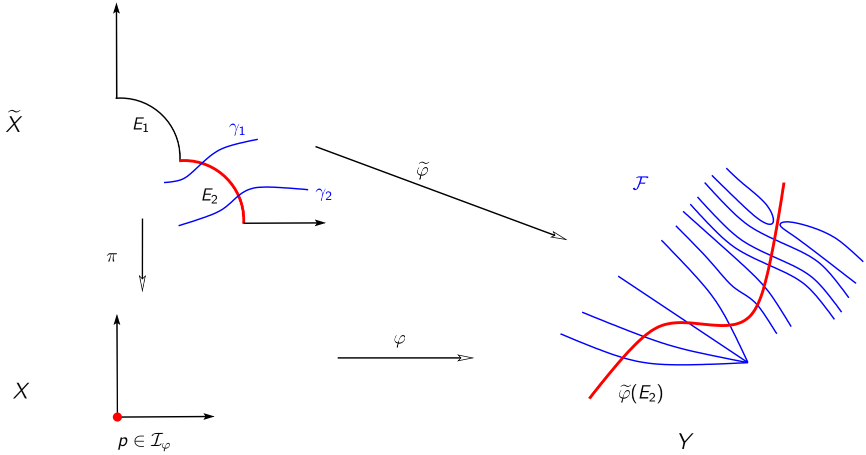
Generation of singularities

Lemma 4 Let $\varphi : X \dashrightarrow Y$ be a dominant rational map of surfaces, let $p \in \mathcal{I}_\varphi$, and suppose $\pi : \widetilde{X} \to X$ and $\widetilde{\varphi} : \widetilde{X} \to Y$ resolve the indeterminacy of φ.
If $\mathcal{F}$ is a foliation on Y and one of the irreducible components of $\widetilde{\varphi}(\pi^{-1}(p))$ is generically transverse to $\mathcal{F}$, then $\varphi^*\mathcal{F}$ is singular at p.



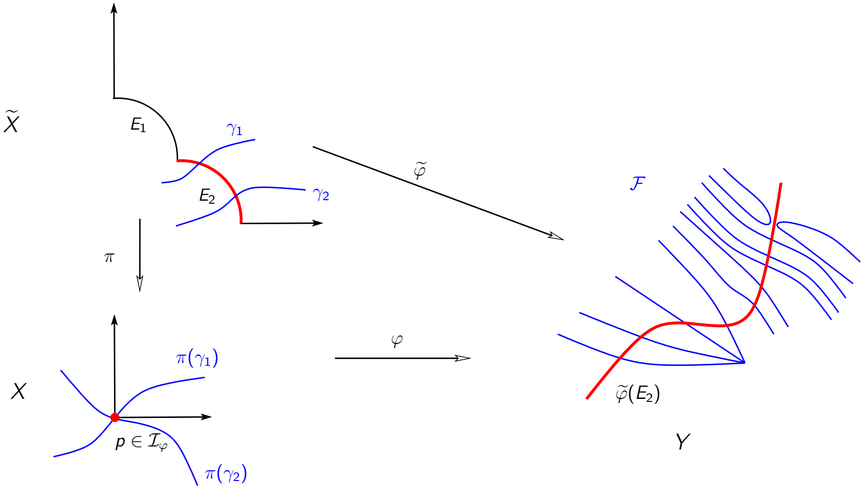
Generation of singularities

Lemma 4 Let $\varphi : X \dashrightarrow Y$ be a dominant rational map of surfaces, let $p \in \mathcal{I}_\varphi$, and suppose $\pi : \widetilde{X} \to X$ and $\widetilde{\varphi} : \widetilde{X} \to Y$ resolve the indeterminacy of φ.
If $\mathcal{F}$ is a foliation on Y and one of the irreducible components of $\widetilde{\varphi}(\pi^{-1}(p))$ is generically transverse to $\mathcal{F}$, then $\varphi^*\mathcal{F}$ is singular at p.



Generation of singularities

Lemma 4 Let $\varphi : X \dashrightarrow Y$ be a dominant rational map of surfaces, let $p \in \mathcal{I}_\varphi$, and suppose $\pi : \widetilde{X} \to X$ and $\widetilde{\varphi} : \widetilde{X} \to Y$ resolve the indeterminacy of φ.
If $\mathcal{F}$ is a foliation on Y and one of the irreducible components of $\widetilde{\varphi}(\pi^{-1}(p))$ is generically transverse to $\mathcal{F}$, then $\varphi^*\mathcal{F}$ is singular at p.



Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \ = \ [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1:0:1]\}$.

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \ = \ [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1:0:1]\}$.

Critical set: $|D\varphi| = -4Y(X-Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\mathrm{coll}} &:= \{X = Z\} \text{ and} \\ L_Y &:= \{Y = 0\}. \end{aligned}$$

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \; = \; [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1 : 0 : 1]\}$.

Critical set: $|D\varphi| = -4Y(X-Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\mathrm{coll}} &:= \{X = Z\} \text{ and} \\ L_Y &:= \{Y = 0\}. \end{aligned}$$

L_{coll} is exceptional and L_Y is not.

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1:0:1]\}$.

Critical set: $|D\varphi| = -4Y(X-Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\rm coll} &:= \{X = Z\} \text{ and} \\ L_Y &:= \{Y = 0\}. \end{aligned}$$

$L_{\rm coll}$ is exceptional and L_Y is not.

$$L_{\rm coll} \setminus \{p\} \longrightarrow [1:0:0] \longrightarrow [0:1:-1] \longrightarrow [-1:0:1] \longrightarrow [0:1:0] \curvearrowright [1:0:0]$$

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] = [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1 : 0 : 1]\}$.

Critical set: $|D\varphi| = -4Y(X-Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\mathrm{coll}} &:= \{X = Z\} \text{ and} \\ L_Y &:= \{Y = 0\}. \end{aligned}$$

L_{coll} is exceptional and L_Y is not.

$$L_{\mathrm{coll}} \setminus \{p\} \longrightarrow [1 : 0 : 0] \longrightarrow [0 : 1 : -1] \longrightarrow [-1 : 0 : 1] \longrightarrow [0 : 1 : 0]$$

Lemma of Fornaess-Sibony gives $\deg_{\mathrm{alg}}(\varphi^n) = (\deg_{\mathrm{alg}}(\varphi))^n = 2^n$, giving $\lambda_1(\varphi) = 2$.

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \;=\; [-Y^2 : X(X-Z) : -(X+Z)(X-Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1 : 0 : 1]\}$.

Critical set: $|D\varphi| = -4Y(X-Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\mathrm{coll}} &:= \{X = Z\} \text{ and} \\ L_Y &:= \{Y = 0\}. \end{aligned}$$

L_{coll} is exceptional and L_Y is not.

$$L_{\mathrm{coll}} \setminus \{p\} \longrightarrow [1:0:0] \longrightarrow [0:1:-1] \longrightarrow [-1:0:1] \longrightarrow [0:1:0] \curvearrowright [1:0:0]$$

Lemma of Fornaess-Sibony gives $\deg_{\mathrm{alg}}(\varphi^n) = (\deg_{\mathrm{alg}}(\varphi))^n = 2^n$, giving $\lambda_1(\varphi) = 2$.

$[1 : 1 : -1]$ is neither a critical value nor in the image of the indeterminacy and $\varphi^{-1}\{[1 : 1 : -1]\} = \{[1 : \pm i : 0]\}$, so $\lambda_2(\varphi) = 2$.

Dynamical degrees of $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$

$$\varphi[X : Y : Z] \; = \; [-Y^2 : X(X - Z) : -(X + Z)(X - Z)].$$

Indeterminacy set: $\mathcal{I}_\varphi := \{p := [1 : 0 : 1]\}$.

Critical set: $|D\varphi| = -4Y(X - Z)^2$, so there are two critical lines:

$$\begin{aligned} L_{\rm coll} \quad &:= \quad \{X = Z\} \text{ and} \\ L_Y \quad &:= \quad \{Y = 0\}. \end{aligned}$$

$L_{\rm coll}$ is exceptional and L_Y is not.

$$L_{\rm coll} \setminus \{p\} \longrightarrow [1 : 0 : 0] \longrightarrow [0 : 1 : -1] \longrightarrow [-1 : 0 : 1] \longrightarrow [0 : 1 : 0]$$

Lemma of Fornaess-Sibony gives $\deg_{\rm alg}(\varphi^n) = (\deg_{\rm alg}(\varphi))^n = 2^n$, giving $\lambda_1(\varphi) = 2$.

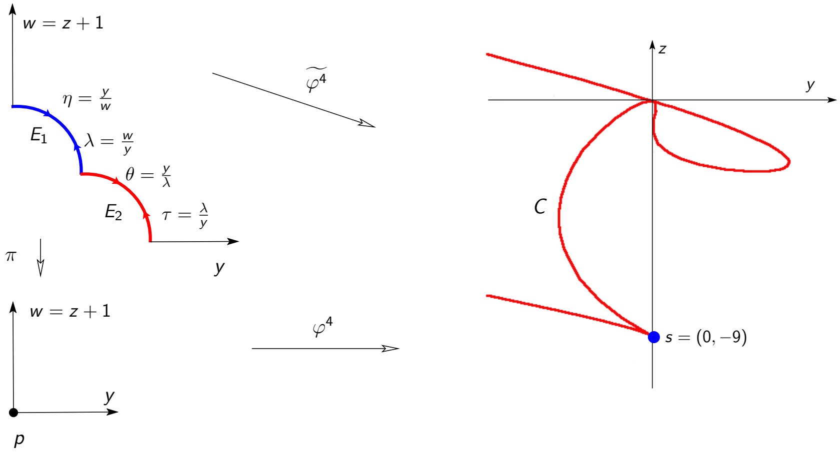
$[1 : 1 : -1]$ is neither a critical value nor in the image of the indeterminacy and $\varphi^{-1}\{[1 : 1 : -1]\} = \{[1 : \pm i : 0]\}$, so $\lambda_2(\varphi) = 2$.

$\lambda_1(\varphi) = 2 = \lambda_2(\varphi)$, so φ is not cohomologically hyperbolic.

Two blow-ups resolve p for φ^4

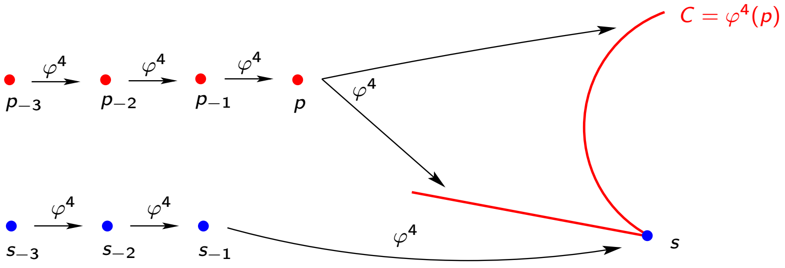
Lemma

The map $\varphi^{\mathbf{4}} : \mathbb{P}^{\mathbf{2}} \dashrightarrow \mathbb{P}^{\mathbf{2}}$ lifts to a rational map $\widetilde{\varphi^{\mathbf{4}}} : \widetilde{\mathbb{P}^{\mathbf{2}}} \dashrightarrow \mathbb{P}^{\mathbf{2}}$ that resolves the indeterminacy of $\varphi^{\mathbf{4}}$ at p. We have that $\widetilde{\varphi^{\mathbf{4}}}(E_{\mathbf{1}}) = [1 : 0 : -9] =: s$ and that $\widetilde{\varphi^{\mathbf{4}}}(E_{\mathbf{2}})$ is an irreducible algebraic curve C of degree 8 that is singular at s.



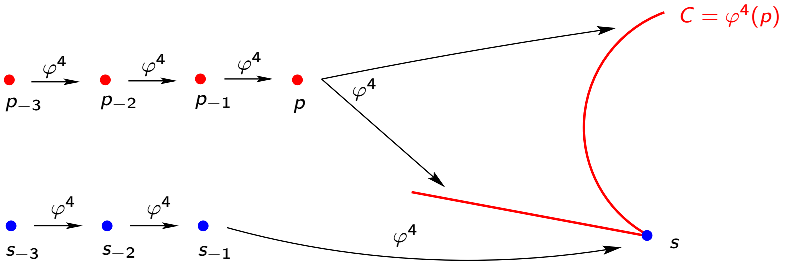
Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



Proof of Theorem 1

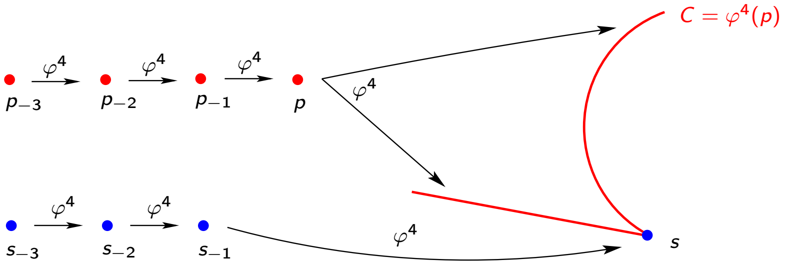
Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



Let $\mathcal{F}$ be any foliation.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

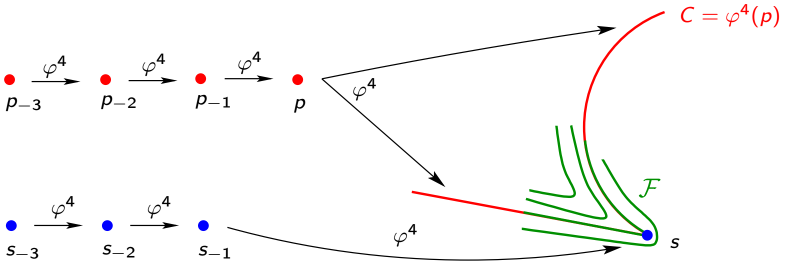


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

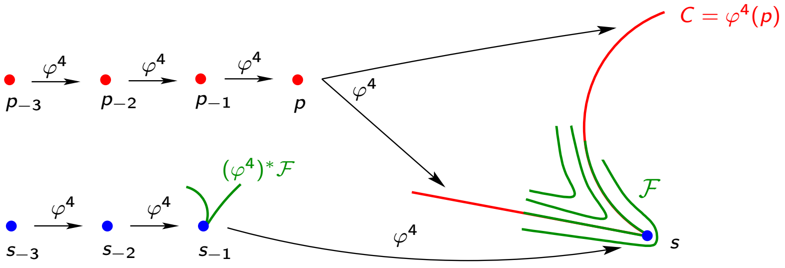


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

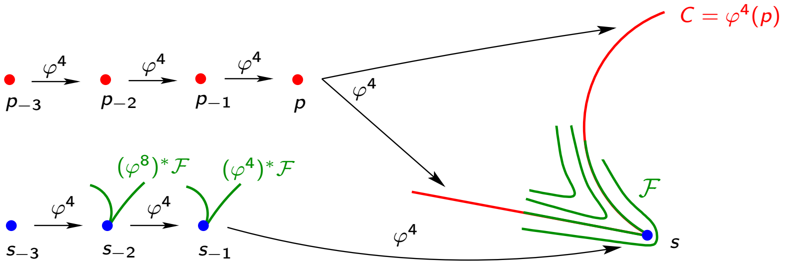


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

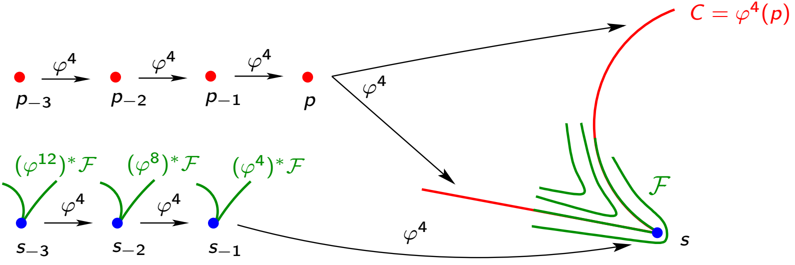


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

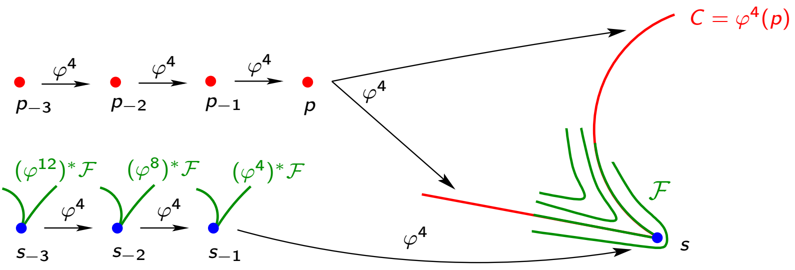


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



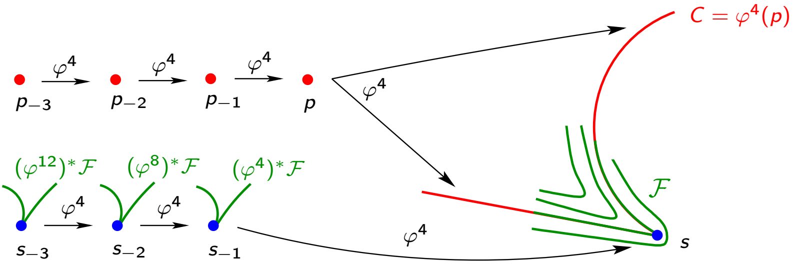
Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

For all n we have that $(\varphi^{4n})^*\mathcal{F}$ is singular at s_{-n}.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



Let $\mathcal{F}$ be any foliation.

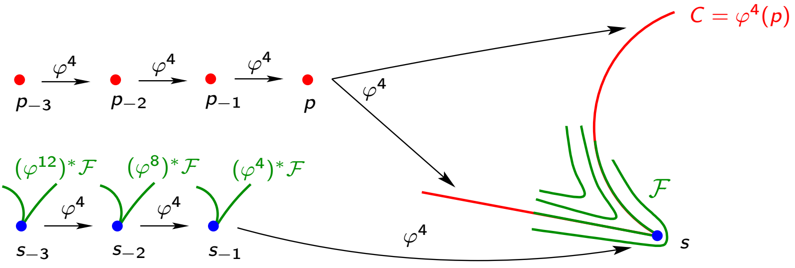
According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

For all n we have that $(\varphi^{4n})^*\mathcal{F}$ is singular at s_{-n}.

If $\varphi^*\mathcal{F} = \mathcal{F}$, then $\mathcal{F}$ is singular at the infinitely many points $\{s_{-i}\}_{i=0}^{\infty}$.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



Let $\mathcal{F}$ be any foliation.

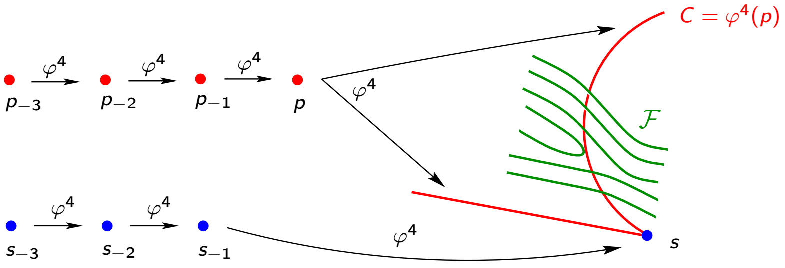
According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 1: C is a leaf of $\mathcal{F}$.

For all n we have that $(\varphi^{4n})^*\mathcal{F}$ is singular at s_{-n}.

If $\varphi^*\mathcal{F} = \mathcal{F}$, then $\mathcal{F}$ is singular at the infinitely many points $\{s_{-i}\}_{i=0}^{\infty}$.
Contradiction.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

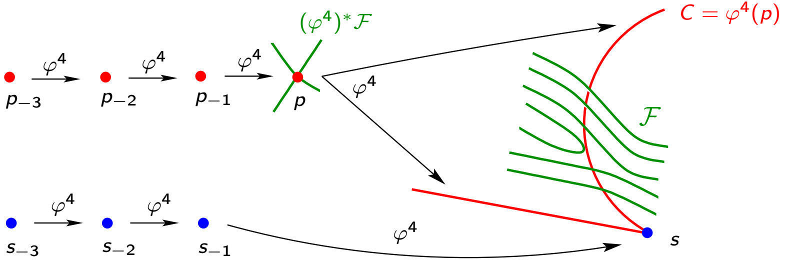


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

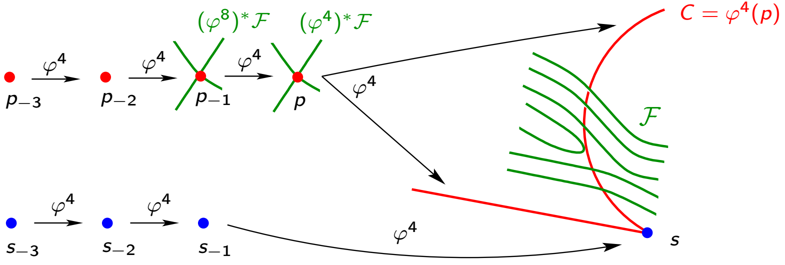


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

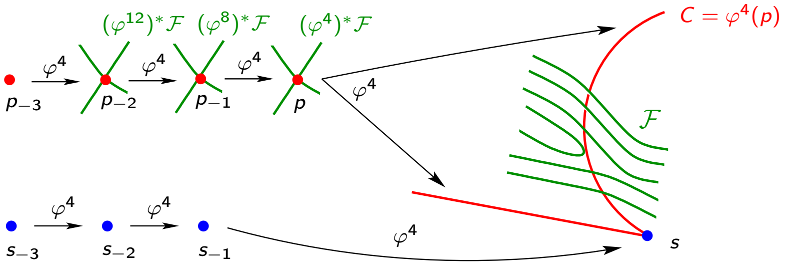


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

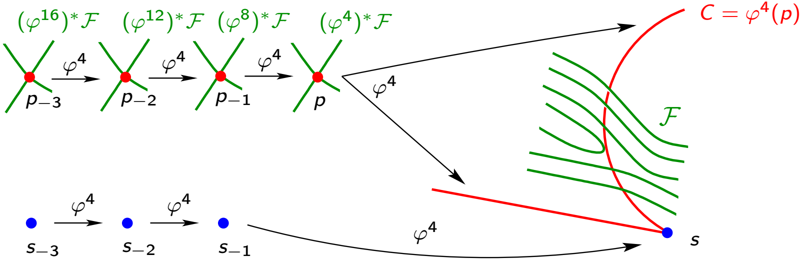


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.

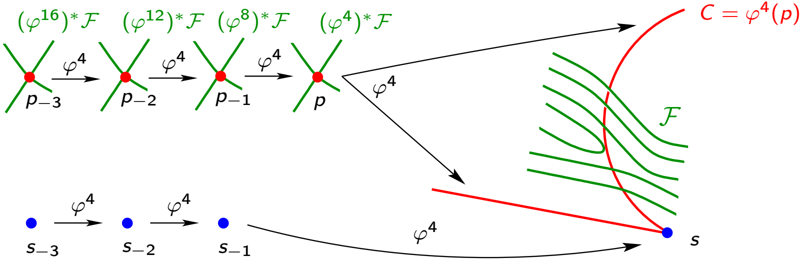


Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

Proof of Theorem 1

Both p and s have preorbits $\{p_{-i}\}_{i=0}^{\infty}$ and $\{s_{-i}\}_{i=0}^{\infty}$ consisting of infinitely many distinct points at each of which φ^4 is holomorphic and finite.



Let $\mathcal{F}$ be any foliation.

According to Lemma 1, C is either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ away from finitely many points. Case 2: C is transverse to $\mathcal{F}$ away from finitely many points.

For all n we have that $(\varphi^{4n})^*\mathcal{F}$ is singular at p_{-n+1}.

If $\varphi^*\mathcal{F} = \mathcal{F}$, then $\mathcal{F}$ is singular at the infinitely many points $\{s_{-n+1}\}_{i=1}^{\infty}$.
Contradiction.

General criteria for a map not to preserve a foliation

General criteria for a map not to preserve a foliation

Theorem 2 (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfies

1. There exists $p \in \mathcal{I}_\eta$ and some iterate k so that η^k blows up p to a singular curve C.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.
3. There is a singular point $s \in C$ having an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

General criteria for a map not to preserve a foliation

Theorem 2 (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfies

1. There exists $p \in \mathcal{I}_\eta$ and some iterate k so that η^k blows up p to a singular curve C.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.
3. There is a singular point $s \in C$ having an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

A theorem of Jouanolou asserts that a foliation $\mathcal{F}$ cannot have algebraic leaves of arbitrarily high degree. This implies the following variant of Theorem 2:

General criteria for a map not to preserve a foliation

Theorem 2 (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfies

1. There exists $p \in \mathcal{I}_\eta$ and some iterate k so that η^k blows up p to a singular curve C.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.
3. There is a singular point $s \in C$ having an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

A theorem of Jouanolou asserts that a foliation $\mathcal{F}$ cannot have algebraic leaves of arbitrarily high degree. This implies the following variant of Theorem 2:

Theorem 2' (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ has an indeterminate point p such that

1. For each n the blow-up $\eta^n(p)$ contains an irreducible curve C_n so that $\limsup \deg(C_n) = \infty$.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

General criteria for a map not to preserve a foliation

Theorem 2 (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfies

1. There exists $p \in \mathcal{I}_\eta$ and some iterate k so that η^k blows up p to a singular curve C.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.
3. There is a singular point $s \in C$ having an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

A theorem of Jouanolou asserts that a foliation $\mathcal{F}$ cannot have algebraic leaves of arbitrarily high degree. This implies the following variant of Theorem 2:

Theorem 2' (Kaschner–Pérez-R) Assume the map $\eta : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ has an indeterminate point p such that

1. For each n the blow-up $\eta^n(p)$ contains an irreducible curve C_n so that $\limsup \deg(C_n) = \infty$.
2. The point p has an infinite preorbit along which η is a finite holomorphic map.

Then no iterate of η preserves a foliation.

Theorem 2' will be crucial in the second part of the talk.

Ergodic properties of maps with equal dynamical degrees

It is easy to construct rational maps with equal dynamical degrees and several distinct measures of maximal entropy.

Ergodic properties of maps with equal dynamical degrees

It is easy to construct rational maps with equal dynamical degrees and several distinct measures of maximal entropy.

Example 1

$$f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 \qquad f(x, y) = (x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$ and an invariant measure of entropy $\log 2$ in each vertical line.

Ergodic properties of maps with equal dynamical degrees

It is easy to construct rational maps with equal dynamical degrees and several distinct measures of maximal entropy.

Example 1

$$f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 \qquad f(x,y) = (x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$ and an invariant measure of entropy $\log 2$ in each vertical line.

Example 2

$$f : \mathbb{CP}^1 \times \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad f(x,y) = (2x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$, a repelling invariant measure of entropy $\log 2$ supported in $\{0\} \times \mathbb{CP}^1$ and a saddle invariant measure supported in $\{\infty\} \times \mathbb{CP}^1$.

Ergodic properties of maps with equal dynamical degrees

It is easy to construct rational maps with equal dynamical degrees and several distinct measures of maximal entropy.

Example 1

$$f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 \qquad f(x, y) = (x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$ and an invariant measure of entropy $\log 2$ in each vertical line.

Example 2

$$f : \mathbb{CP}^1 \times \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad f(x, y) = (2x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$, a repelling invariant measure of entropy $\log 2$ supported in $\{0\} \times \mathbb{CP}^1$ and a saddle invariant measure supported in $\{\infty\} \times \mathbb{CP}^1$.

Both of these have the measures of maximal entropy each supported in a hypersurface (so, it's cheating).

Ergodic properties of maps with equal dynamical degrees

It is easy to construct rational maps with equal dynamical degrees and several distinct measures of maximal entropy.

Example 1

$$f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2 \qquad f(x, y) = (x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$ and an invariant measure of entropy $\log 2$ in each vertical line.

Example 2

$$f : \mathbb{CP}^1 \times \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1 \qquad f(x, y) = (2x, y^2)$$

has $\lambda_1(f) = \lambda_2(f) = 2$, a repelling invariant measure of entropy $\log 2$ supported in $\{0\} \times \mathbb{CP}^1$ and a saddle invariant measure supported in $\{\infty\} \times \mathbb{CP}^1$.

Both of these have the measures of maximal entropy each supported in a hypersurface (so, it's cheating).

Moreover, both are skew products (so, they're boring).

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$

$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$

$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$

$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L.

For $t \in [0, 1]$, let $\delta_t = t\delta + (1 - t)\mathrm{id}$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L.

For $t \in [0, 1]$, let $\delta_t = t\delta + (1 - t)\mathrm{id}$.

$f_t := \delta_t \circ f$ for any $t \in (0, 1)$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\mathrm{alg}}(f) = 4 = \deg_{\mathrm{top}}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L.

For $t \in [0, 1]$, let $\delta_t = t\delta + (1 - t)\mathrm{id}$.

$f_t := \delta_t \circ f$ for any $t \in (0, 1)$.

Has $\lambda_1(f_t) = 4 = \lambda_2(f_t)$ and there is still a measure of entropy $\log 4$ supported on a Cantor set $\mathcal{K}$.

Perturbed Chebyshev Map

Two-dimensional Chebyshev map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ is given by $f = s \circ q$ where

$$s[x_1 : x_2 : x_3] = [x_1^2 : x_2^2 : x_3^2], \quad \text{and}$$
$$q[x_1 : x_2 : x_3] = [x_1(-x_1 + x_2 + x_3) : x_2(x_1 - x_2 + x_3) : x_3(x_1 + x_2 - x_3)].$$

$\deg_{\rm alg}(f) = 4 = \deg_{\rm top}(f)$. (However, $\lambda_1(f) = 2 < \lambda_2(f) = 4$.)

Has measure of maximal entropy on real slice that is absolutely continuous with respect to Lebesgue. Supported inside a circle centered at $[1 : 1 : 1]$.

Perturbed Chebyshev map

Let $L = \{x_1 + x_2 + x_3 = 0\}$ and $p_0 = [1 : 1 : 1]$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L.

For $t \in [0, 1]$, let $\delta_t = t\delta + (1 - t)\mathrm{id}$.

$f_t := \delta_t \circ f$ for any $t \in (0, 1)$.

Has $\lambda_1(f_t) = 4 = \lambda_2(f_t)$ and there is still a measure of entropy $\log 4$ supported on a Cantor set $\mathcal{K}$.

Han Liu studied f and f_t in his 2014 Ph.D. Thesis. (Student of Diller.)

Perturbed Chebyshev Map

Building off of Han's work:

Perturbed Chebyshev Map

Building off of Han's work:

Theorem 3 (Diller-R) For any $t \in (0, 1)$ that is sufficiently close to 1:

1. the dynamical degrees of $f_t : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfy $\lambda_1(f_t) = \lambda_2(f_t) = 4$,

Perturbed Chebyshev Map

Building off of Han's work:

Theorem 3 (Diller-R) For any $t \in (0, 1)$ that is sufficiently close to 1:

1. the dynamical degrees of $f_t : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfy $\lambda_1(f_t) = \lambda_2(f_t) = 4$,
2. f_t has no invariant foliation on $\mathbb{CP}^2$,

Perturbed Chebyshev Map

Building off of Han's work:

Theorem 3 (Diller-R) For any $t \in (0,1)$ that is sufficiently close to 1:

1. the dynamical degrees of $f_t : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfy $\lambda_1(f_t) = \lambda_2(f_t) = 4$,
2. f_t has no invariant foliation on $\mathbb{CP}^2$,
3. f_t has an invariant repelling Cantor set $\mathcal{K}$, an invariant hyperbolic saddle set Λ, and three attracting fixed points e_1, e_2, and e_3. Any point z not in the iterated indeterminacy set $\mathcal{I}_\infty := \bigcup \mathcal{I}_{f_t^n}$ satisfies

$$z \in \mathcal{K} \cup \mathcal{W}^s(\Lambda) \cup \bigcup_{i=1}^{3} \mathcal{W}^s(e_i). \tag{1}$$

Perturbed Chebyshev Map

Building off of Han's work:

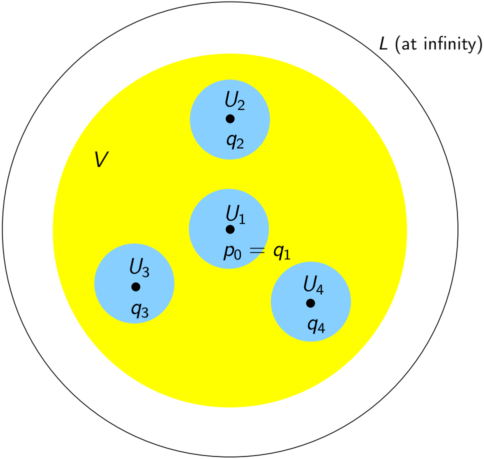
Theorem 3 (Diller-R) For any $t \in (0,1)$ that is sufficiently close to 1:

1. the dynamical degrees of $f_t : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ satisfy $\lambda_1(f_t) = \lambda_2(f_t) = 4$,
2. f_t has no invariant foliation on $\mathbb{CP}^2$,
3. f_t has an invariant repelling Cantor set $\mathcal{K}$, an invariant hyperbolic saddle set Λ, and three attracting fixed points e_1, e_2, and e_3. Any point z not in the iterated indeterminacy set $\mathcal{I}_\infty := \bigcup \mathcal{I}_{f_t^n}$ satisfies

$$z \in \mathcal{K} \cup \mathcal{W}^s(\Lambda) \cup \bigcup_{i=1}^{3} \mathcal{W}^s(e_i). \tag{1}$$

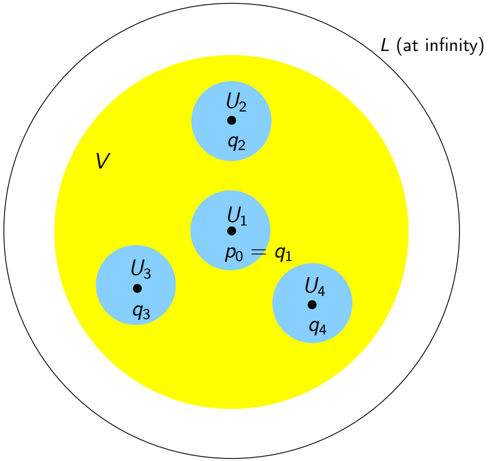
4. f_t has two distinct ergodic invariant measures μ and ν of maximal entropy $\log 4$:
   (a) $\operatorname{supp} \mu = \mathcal{K}$, so that it is repelling, and $\operatorname{supp} \nu = \Lambda$ so that it is of saddle-type,
   (b) neither measure is supported in an algebra curve.

Repelling Measure



$f^{-1}(p_0) = \{q_1, q_2, q_3, q_4\}$. For t close enough to 1, $V \Subset f_t(U_i)$.

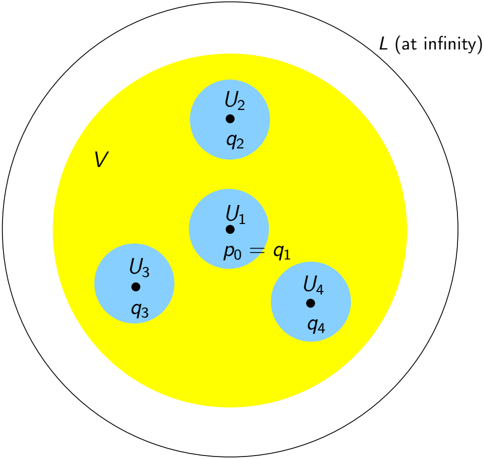
Repelling Measure



$f^{-1}(p_0) = \{q_1, q_2, q_3, q_4\}$. For t close enough to 1, $V \Subset f_t(U_i)$.

$\mathcal{K} = \bigcap f^{-n}(V)$ is a Cantor set.

Repelling Measure



$f^{-1}(p_0) = \{q_1, q_2, q_3, q_4\}$. For t close enough to 1, $V \Subset f_t(U_i)$.

$\mathcal{K} = \bigcap f^{-n}(V)$ is a Cantor set.

μ is pullback of Bournoulli measure from full four shift.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i \le 0} \,:\, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i \le 0}) = (f(x_i))_{i \le 0}.$$

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i\leq 0} \,:\, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i\leq 0}) = (f(x_i))_{i\leq 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i\leq 0} \,:\, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i\leq 0}) = (f(x_i))_{i\leq 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$. Also of entropy $\log 4$.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i \le 0} \, : \, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i \le 0}) = (f(x_i))_{i \le 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$. Also of entropy $\log 4$.

For t sufficiently close to 1, one can construct a map $h_t : \widehat{\Lambda_1} \to \mathbb{CP}^2$ so that

$$\begin{array}{ccc} \widehat{\Lambda_1} & \xrightarrow{\widehat{f_1}} & \widehat{\Lambda_1} \\ \downarrow h_t & & \downarrow \widehat{h_t} \\ h_t\left(\widehat{\Lambda_1}\right) & \xrightarrow{f_t} & h_t\left(\widehat{\Lambda_1}\right). \end{array}$$

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i \leq 0} \, : \, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i \leq 0}) = (f(x_i))_{i \leq 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$. Also of entropy $\log 4$.

For t sufficiently close to 1, one can construct a map $h_t : \widehat{\Lambda_1} \to \mathbb{CP}^2$ so that

$$\begin{array}{ccc} \widehat{\Lambda_1} & \xrightarrow{\widehat{f_1}} & \widehat{\Lambda_1} \\ \Big\downarrow h_t & & \Big\downarrow \widehat{h_t} \\ h_t\left(\widehat{\Lambda_1}\right) & \xrightarrow{f_t} & h_t\left(\widehat{\Lambda_1}\right). \end{array}$$

$\Lambda_t := h_t\left(\widehat{\Lambda_1}\right)$ is a saddle set for f_t.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i \le 0} \,:\, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i \le 0}) = (f(x_i))_{i \le 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$. Also of entropy $\log 4$.

For t sufficiently close to 1, one can construct a map $h_t : \widehat{\Lambda_1} \to \mathbb{CP}^2$ so that

$$\begin{array}{ccc} \widehat{\Lambda_1} & \xrightarrow{\widehat{f_1}} & \widehat{\Lambda_1} \\ \downarrow{\scriptstyle h_t} & & \downarrow{\scriptstyle \widehat{h_t}} \\ h_t\left(\widehat{\Lambda_1}\right) & \xrightarrow{f_t} & h_t\left(\widehat{\Lambda_1}\right). \end{array}$$

$\Lambda_t := h_t\left(\widehat{\Lambda_1}\right)$ is a saddle set for f_t.

Using some details about the geometry of $f_t(L)$ and Λ_1 we can show that $h_t : \widehat{\Lambda_1} \to \Lambda_t$ is a conjugacy.

Saddle Measure

One can check that $f_1|_L : L \to L$ is hyperbolic. Let Λ_1 be its Julia set.

Natural extension:

$$\widehat{f_1} : \widehat{\Lambda_1} \to \widehat{\Lambda_1} \quad \text{where,} \quad \widehat{\Lambda_1} := \{(x_i)_{i \leq 0} \,:\, x_i \in \Lambda \text{ and } f_1(x_i) = x_{i+1}\}, \quad \text{and}$$

$$\widehat{f_1}((x_i)_{i\leq 0}) = (f(x_i))_{i \leq 0}.$$

Measure of maximal entropy ν_1 for $f_1|_L$ lifts to an invariant measure $\widehat{\nu_1}$ for $\widehat{f_1}$. Also of entropy $\log 4$.

For t sufficiently close to 1, one can construct a map $h_t : \widehat{\Lambda_1} \to \mathbb{CP}^2$ so that

$$\begin{array}{ccc} \widehat{\Lambda_1} & \xrightarrow{\widehat{f_1}} & \widehat{\Lambda_1} \\ \Big\downarrow{\scriptstyle h_t} & & \Big\downarrow{\scriptstyle \widehat{h_t}} \\ h_t\left(\widehat{\Lambda_1}\right) & \xrightarrow{f_t} & h_t\left(\widehat{\Lambda_1}\right). \end{array}$$

$\Lambda_t := h_t\left(\widehat{\Lambda_1}\right)$ is a saddle set for f_t.

Using some details about the geometry of $f_t(L)$ and Λ_1 we can show that $h_t : \widehat{\Lambda_1} \to \Lambda_t$ is a conjugacy.

Let $\nu_t := h_{t*}(\widehat{\nu_1})$ so that ν_t has entropy $\log 4$.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

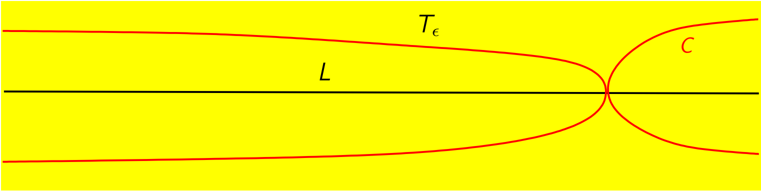
$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:

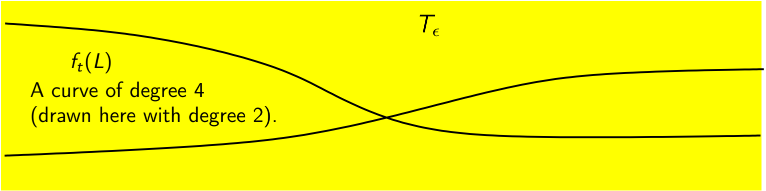


No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:

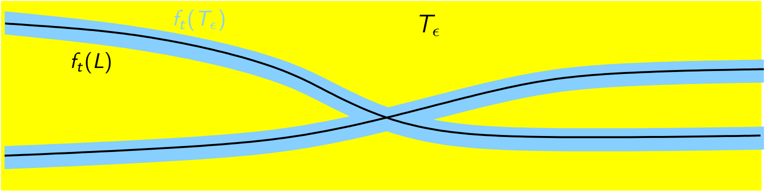


No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:

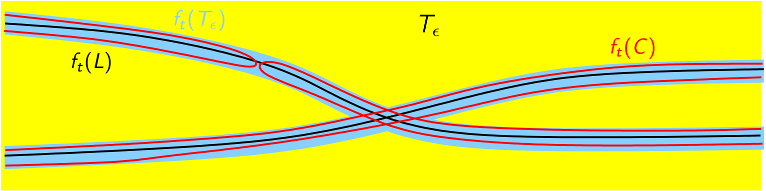


No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:

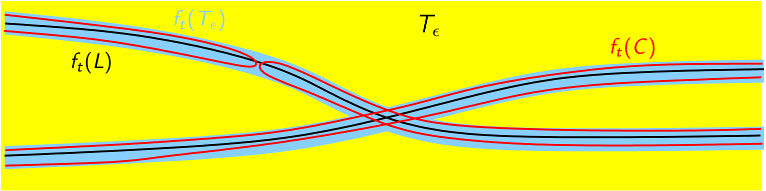


No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



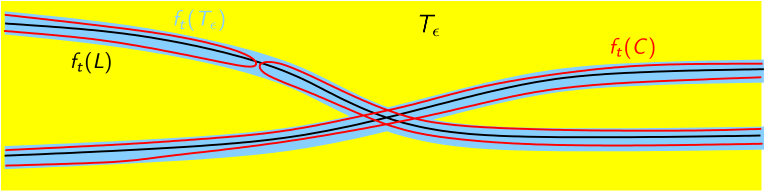
No invariant foliation:

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



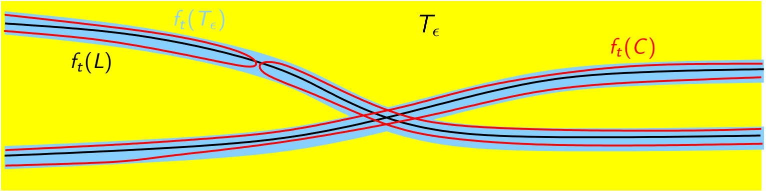
No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

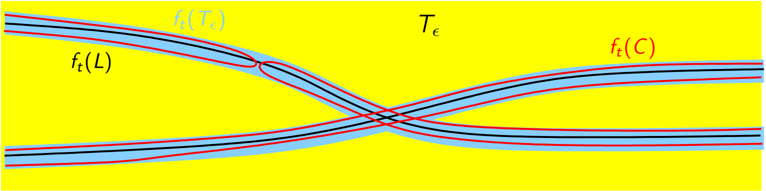
For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

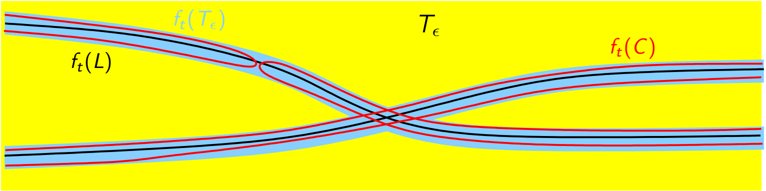
Theorem 2' $\Rightarrow$ f_t doesn't preserve a foliation.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

Theorem 2' $\Rightarrow$ f_t doesn't preserve a foliation.

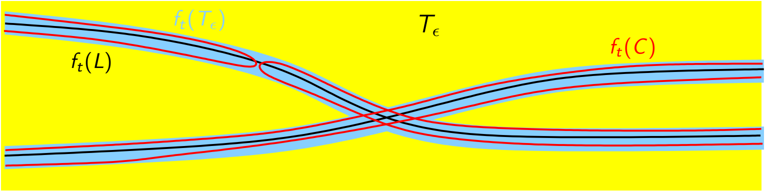
ν_t not supported on a curve:

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

Theorem 2' $\Rightarrow$ f_t doesn't preserve a foliation.

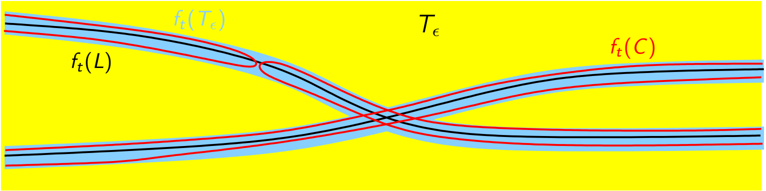
ν_t not supported on a curve:
$\mathrm{supp}\ \nu_t \subset T_\epsilon$.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

Theorem 2' $\Rightarrow$ f_t doesn't preserve a foliation.

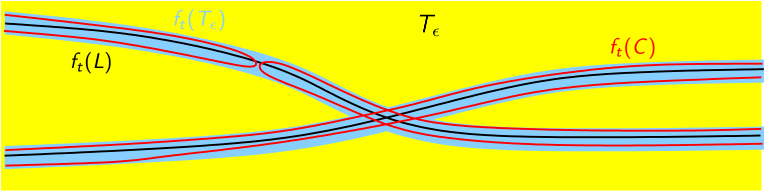
ν_t not supported on a curve:
$\mathrm{supp}\ \nu_t \subset T_\epsilon$. If $\mathrm{supp}\ \nu_t \subset C$, then C is invariant and $C \subset T_\epsilon$.

No invariant foliation & saddle measure ν_t not supported in a curve

Lemma Let T_ϵ be the tube of radius ϵ around L. For any t sufficiently close to 1, $f_t(T_\epsilon) \Subset T_\epsilon$ and for any irreducible algebraic curve $C \subset T_\epsilon$ we have

$$\deg(f_t(C)) = 4 \cdot \deg(C).$$

Illustration of the proof:



No invariant foliation: For t sufficiently close to 1, any indeterminate point for f_t will blow-up to a curve in T_ϵ. Lemma $\Rightarrow$ further iterates grow in degree.

For t is close enough to 1, any exceptional curves are mapped into T_ϵ so that indeterminate points have infinite preorbits along which f_t is finite.

Theorem 2' $\Rightarrow$ f_t doesn't preserve a foliation.

ν_t not supported on a curve:
$\text{supp}\ \nu_t \subset T_\epsilon$. If $\text{supp}\ \nu_t \subset C$, then C is invariant and $C \subset T_\epsilon$.
However, the Lemma prohibits invariant curves in T_ϵ.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

Choose a point $p_0 \in \mathbb{CP}^2$ outside $L \cup f(L) \cup f(\mathcal{I}(f)) \cup f(\mathcal{C}(f))$.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

Choose a point $p_0 \in \mathbb{CP}^2$ outside $L \cup f(L) \cup f(\mathcal{I}(f)) \cup f(\mathcal{C}(f))$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L and let $\delta_t = t\delta + (1-t)\mathrm{id}$ for $t \in (0,1)$.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

Choose a point $p_0 \in \mathbb{CP}^2$ outside $L \cup f(L) \cup f(\mathcal{I}(f)) \cup f(\mathcal{C}(f))$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L and let $\delta_t = t\delta + (1-t)\mathrm{id}$ for $t \in (0,1)$.

For t close enough to 1 we have

1. $f_t := \delta_t \circ f$ satisfies $\lambda_1(f) = a^2 = \lambda_2(f)$;
2. f_t has at least two distinct measures of maximal entropy with disjoint support.
3. No iterate of f_t preserves a foliation.
4. Neither measure is supported on an algebraic curve.

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

Choose a point $p_0 \in \mathbb{CP}^2$ outside $L \cup f(L) \cup f(\mathcal{I}(f)) \cup f(\mathcal{C}(f))$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L and let $\delta_t = t\delta + (1-t)\mathrm{id}$ for $t \in (0,1)$.

For t close enough to 1 we have

1. $f_t := \delta_t \circ f$ satisfies $\lambda_1(f) = a^2 = \lambda_2(f)$;
2. f_t has at least two distinct measures of maximal entropy with disjoint support.
3. No iterate of f_t preserves a foliation.
4. Neither measure is supported on an algebraic curve.

Hyperbolic theory for $f_1|_L$ replaced with application of results from:
De Thélin and Vigny *Entropy of meromorphic maps and dynamics of birational maps*. Mém. SMF. No. 122 (2010)

General Construction

Theorem 4 (Diller-R) Let $g : \mathbb{CP}^2 \to \mathbb{CP}^2$ be an endomorphism and $h : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ a birational map, both of algebraic degree $a \geq 2$, and suppose $h(\mathcal{E}(h)) \notin \mathcal{C}(g)$.
Let $f = g \circ h$.

Fix a line $L \subset \mathbb{CP}^2 \setminus \mathcal{I}(f)$ that is not contained in $\mathcal{C}(f)$ and such that $f(L)$ is an algebraic curve of degree a^2.

Choose a point $p_0 \in \mathbb{CP}^2$ outside $L \cup f(L) \cup f(\mathcal{I}(f)) \cup f(\mathcal{C}(f))$.

Let $\delta_1 : \mathbb{CP}^2 \setminus \{p_0\} \to L$ be the central projection from p_0 to L and let $\delta_t = t\delta + (1-t)\mathrm{id}$ for $t \in (0,1)$.

For t close enough to 1 we have

1. $f_t := \delta_t \circ f$ satisfies $\lambda_1(f) = a^2 = \lambda_2(f)$;
2. f_t has at least two distinct measures of maximal entropy with disjoint support.
3. No iterate of f_t preserves a foliation.
4. Neither measure is supported on an algebraic curve.

Hyperbolic theory for $f_1|_L$ replaced with application of results from:
De Thélin and Vigny *Entropy of meromorphic maps and dynamics of birational maps*. Mém. SMF. No. 122 (2010)

Rest of the proof is essentially the same.

Question

Does a "generic" rational map $f : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$ with $\deg_{\mathrm{alg}}(f) = \deg_{\mathrm{top}}(f)$ satisfy:

1. No iterate of f preserves a foliation, and
2. f has two distinct measures of maximal entropy, neither of which is supported on a curve?

Happy Birthday Hamal!