On the postcritical set of a rational map

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Abstract

The postcritical set $P(f)$ of a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is the smallest forward invariant subset of $\mathbb{P}^1$ that contains the critical values of f. In this paper we show that every finite set $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ can be realized as the postcritical set of a rational map. We also show that every map $F : X \to X$ defined on a finite set $X \subset \mathbb{P}^1(\mathbb{C})$ can be realized by a rational map $f : P(f) \to P(f)$, provided we allow small perturbations of the set X. The proofs involve Belyi's theorem and iteration on Teichmüller space.

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1 Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$. Let $C(f) \subset \mathbb{P}^1$ denote the set of critical points of f, and let $V(f)$ denote the set of critical values. The *postcritical set* of f is defined by

$$P(f) = \bigcup_{n \geq 0} f^n(V(f)).$$

A rational map is *postcritically finite* if $|P(f)| < \infty$. Most postcritically finite rational maps are *rigid*, in the sense defined in §2; for example, f is rigid if $|P(f)| > 4$. When f is rigid, it is conformally conjugate to a rational map g defined over the algebraic closure of $\mathbb{Q}$, in which case we have $P(g) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$. In this paper we prove the converse:

Theorem 1.1 *For any finite set $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ with $|X| \geq 2$, there exists a rigid, postcritically finite rational map such that $P(f) = X$.*

We can also arrange that $P(f) \subset C(f)$, which implies that f is hyperbolic. The proof (§2) uses Belyi's theorem; consequently, the degree of the map f we construct may be enormous, even when the set X is rather simple.

In light of Theorem 1.1, we formulate the following more precise question about dynamics on the postcritical set.

Question 1.2 *Let $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be a finite set. Is every map $F : X \to X$ realized by a rigid rational map $f : P(f) \to P(f)$ with $P(f) = X$?*

It is easy to see the answer can be *no* when $|X| = 2$, and it is *yes* when $|X| = 3$ (for a proof, see §7). We do not know of any negative answer when $|X| = 4$, but most cases are simply open. For example, we do not know the answer for $X = \{0, 1, 4, \infty\}$ and $F(x) = x$.

We can, however, show that Question 1.2 has a positive answer if we allow perturbations of X as a subset of $\mathbb{P}^1(\mathbb{C})$.

Theorem 1.3 *Let $F : X \to X$ be an arbitrary map defined on a finite set $X \subset \mathbb{P}^1(\mathbb{C})$ with $|X| \geq 3$. Then there exists a sequence of rigid postcritically finite rational maps f_n such that $|P(f_n)| = |X|$,*

$$P(f_n) \to X \quad \textit{and} \quad f_n | P(f_n) \to F | X$$

as $n \to \infty$.

The proof (§4) uses iteration on Teichmüller space, as in the proof of Thurston's topological characterization of postcritically finite rational maps [DH].

In §5, we establish the following new Hurwitz-type result, which may be of interest in its own right:

> *Any collection of partitions $\mathcal{P}$ with $|\mathcal{P}| \geq 3$ can be extended to the passport $\mathcal{Q}$ of a rational map, with $|\mathcal{Q}| = |\mathcal{P}|$.*

See Theorem 5.1. (Here the *passport* of a rational map f is the collection of partitions of $\deg(f)$ arising from the fibers over its critical values.) We use this result to strengthen Theorem 1.3 in §6 by showing that we can also specify the multiplicity of F at each point of X.

Other constructions of rational maps with specified postcritical sets are presented in §7.

2 Every finite set of algebraic numbers is a postcritical set

In this section we prove that any finite set $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ with $2 \leq |X| < \infty$ arises as the postcritical set of a rigid rational map (Theorem 1.1). The result for rational maps follows easily from the following variant for polynomials.

Theorem 2.1 *For any finite set of algebraic numbers $X \subset \mathbb{C}$ with $|X| \geq 1$, there exists a polynomial f such that $P(f) \cap \mathbb{C} = X$.*

Polynomials. To begin the proof, we remark that it is easy to construct polynomials with prescribed critical values. It is convenient, when discussing a polynomial f, to omit the point and infinity and let $C_0(f) = C(f) \cap \mathbb{C}$, and similarly for $V_0(f)$ and $P_0(f)$.

Lemma 2.2 *For any finite set $X \subset \mathbb{C}$, there exists a polynomial g with $V_0(g) = X$.*

Proof. There are many ways to prove this result; for example, by induction on $|X|$, using the fact that $V_0(f \circ g) = V_0(f) \cup f(V_0(g))$ and $V_0(z^2 + a) = \{a\}$. ■

(For a more precise result, see Corollary 5.4.)

Let us say β is a *Belyi polynomial* if $V_0(\beta) \subset \{0,1\}$. We will also use following result from [Bel]:

Theorem 2.3 *For any finite set $X \subset \overline{\mathbb{Q}}$, there exists a Belyi polynomial such that $\beta(X) \subset \{0,1\}$.*

Proof of Theorem 2.1. Our aim is to construct a polynomial f such that $P_0(f) = X$. This is easy if $|X| \le 1$, so assume $|X| \ge 2$. Using Lemma 2.2, choose a polynomial g such that $V_0(g) = X$, and precompose with an affine transformation so that $\{0,1\} \subset C_0(g)$. Let β be a Belyi polynomial such that $\beta(X) \subset \{0,1\}$. Finally, let $f = g \circ \beta$.

We claim that $V_0(f) = X$. Indeed, we have

$$V_0(f) = V_0(g \circ \beta) = V_0(g) \cup g(V_0(\beta)) = X \cup g(V_0(\beta));$$

but $V_0(\beta) \subset \{0,1\} \subset C_0(g)$, and therefore $g(V_0(\beta)) \subset X$, so $V_0(f) = X$. In particular $X \subset P_0(f)$. But in fact $P_0(f) = X$, since

$$f(X) = g \circ \beta(X) \subset g(\{0,1\}) \subset X.$$

■

Hyperbolicity. Note that

$$P_0(f) = X \subset \beta^{-1}(C_0(g)) \subset C_0(g \circ \beta) = C_0(f)$$

in the construction above. Thus every periodic point in $P(f)$ is superattracting, and hence f is hyperbolic.

Rigidity. To deduce Theorem 1.1, we must first briefly discuss rigidity. A postcritically finite map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is *rigid* if any postcritically finite rational map g uniformly close enough to f, and with $|P(g)| = |P(f)|$, is in fact conformally conjugate to f. For any fixed d and n, the rigid maps with $\deg(f) = d$ and $|P(f)| = n$ fall into finitely many conjugacy classes.

By a theorem of Thurston [DH], the only postcritically finite rational maps f that are *not* rigid are the flexible Lattès examples, which arise from the addition law on an elliptic curve [Mil2]. These flexible maps have $|P(f)| = 4$ and Julia set $J(f) = \mathbb{P}^1$. Consequently, any postcritically finite rational map with a periodic critical point (such as a polynomial) is rigid.

Proof of Theorem 1.1. Let $X \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be a finite set with $|X| \ge 2$. After a change of coordinates defined over $\overline{\mathbb{Q}}$, we can assume that $\infty \in X$. Then by Theorem 2.1, there exists a polynomial f with $P(f) = X$; and as we have just observed, any postcritically finite polynomial is rigid. ■

Belyi degree and postcritical degree. Given a finite set $X \subset \overline{\mathbb{Q}}$, let $B(X)$ denote the minimum of $\deg(\beta)$ over all Belyi polynomials with $\beta(X) \subset \{0,1\}$; and let $D(X)$ denote the minimum of $\deg(f)$ over all polynomials with $P_0(f) = X$.

Little is known about the general behavior of these degree functions (in particular, lower bounds seem hard to come by); however, the proof of Theorem 2.1 in concert with Corollary 5.4 gives the relation:

$$D(X) \leq B(X) + |X| + 1.$$

Both degree functions seem to merit further study.

3 Contraction on Teichmüller space

The rational maps f constructed in the proof of Theorem 1.1 all satisfy $|f(P(f))| \leq 3$. We next address the problem of realizing more general dynamics on $P(f)$. Our construction will use iteration on Teichmüller space as in [DH]. This section gives the needed background; for more details see [BCT], [Hub].

Teichmüller spaces. Given a finite set $A \subset \mathbb{P}^1$ with $|A| = n$, we let $\mathcal{T}_A \cong \mathcal{T}_{0,n}$ denote the *Teichmüller space* of genus zero Riemann surfaces *marked* by $(\mathbb{P}^1, A)$.

A point in $\mathcal{T}_A$ is specified by another pair $(\mathbb{P}^1, A')$ together with an orientation–preserving *marking* homeomorphism:

$$\phi : (\mathbb{P}^1, A) \to (\mathbb{P}^1, A').$$

Since $A' = \phi(A)$, the marking ϕ alone determines a point $[\phi] \in \mathcal{T}_A$. Two markings ϕ_1, ϕ_2 determine the same point iff we can write $\phi_2 = \alpha \circ \phi_1 \circ \psi$, where $\alpha \in \operatorname{Aut}(\mathbb{P}^1)$ and ψ is isotopic to the identity *rel* A.

The cotangent space to $\mathcal{T}_A$ at $(\mathbb{P}^1, A')$ is naturally identified with the vector space $Q(\mathbb{P}^1 - A')$ consisting of meromorphic differentials $q = q(z)\,dz^2$ on $\mathbb{P}^1$ with at worst simple poles on A' and elsewhere holomorphic. The Teichmüller metric corresponds to the norm

$$\|q\| = \int_{\mathbb{P}^1} |q|$$

on the cotangent space.

Pullback. Now let $F : \mathbb{P}^1 \to \mathbb{P}^1$ be a *smooth* branched covering map with $\deg(F) \geq 2$. The sets $C(F)$, $V(F)$ and $P(F)$ are defined just as for a rational map.

Consider a pair finite sets A and B in $\mathbb{P}^1$ such that

$$F(A) \cup V(F) \subset B.$$

We then have a map of pairs

$$F : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$$

that is branched only over B. By pullback of complex structures, we then obtain a holomorphic map

$$\sigma_F : \mathcal{T}_B \to \mathcal{T}_A.$$

Contraction. Let $\sigma : \mathcal{T}_B \to \mathcal{T}_A$ be a holomorphic map between Teichmüller spaces. By the Schwarz lemma, $\|D\sigma\| \leq 1$ in the Teichmüller metric. We say σ is *contractive* if $\|D\sigma\| < 1$ at every point of $\mathcal{T}_B$. (We also say σ is contractive if $|B| = 3$, since then its image is a single point.) If $A = B$ and σ is contractive, then σ has at most one fixed point.

A branched covering F is *contractive* if σ_F is contractive.

The following result is well known and was a key step in the proof of Thurston's rigidity theorem for postcritically finite rational maps; see [DH, Prop. 3.3].

Proposition 3.1 *The map F is contractive if and only if there is no 4-tuple $B_0 \subset B$ such that*

$$F^{-1}(B_0) \subset A \cup C(F). \tag{3.1}$$

Proof. We may assume $|B| \geq 4$. Suppose $\|D\sigma_F\| = 1$ at some point in $\mathcal{T}_B$. The coderivative of σ_F at this point is given explicitly by a pushforward map of the form

$$f_* : Q(\mathbb{P}^1 - A') \to Q(\mathbb{P}^1 - B'),$$

where f is a rational map of the same topological type as F. Since the domain of f_* is finite dimensional, there exists a nonzero $q \in Q(\mathbb{P}^1 - A')$ such that $\|f_* q\| = \|q\|$. The fact that there is no cancellation under pushforward implies that q is a locally a positive real multiple of the pullback of $f_* q$. In fact, since $\|f^* f_* q\| = \deg(f)\, \|f_* q\| = \|q\|$, we must have

$$f^* f_* q = \deg(f)\, q.$$

Now recall that any meromorphic quadratic differential on $\mathbb{P}^1$ has at least 4 poles. Choose 4 points $B_0 \subset B$ such that $\phi(B_0) \subset B'$ is contained in the poles of $f_* q$. Then the equation above implies that the poles of $f^* f_* q$ lie in A'. Thus any point in $F^{-1}(B_0)$ that does not lie in A must be a critical point of F, giving condition (3.1) above.

For the converse, suppose we have 4 points $B_0 \subset B$ satisfying (3.1). Consider, at any point in $\mathcal{T}_B$, a quadratic differential q with poles only at the 4 points marked by B_0. Then $f^* q$ has poles only at points marked by A, and hence it represents a cotangent vector to $\mathcal{T}_A$. Since $\|f_*(f^* q)\| = \deg(f)\|q\| = \|f^* q\|$, we have $\|D\sigma_F\| = 1$ at every point in $\mathcal{T}_B$. ∎

Example: dynamics. Let f be a rational map with $|P(f)| = 3$. Note that a rational map is a special case of a smooth branched covering. Consider a map of pairs

$$f^k : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B),$$

such that $f^k(A) \cup P(f) \subset B$.

Proposition 3.2 *The pullback map*

$$\sigma_{f^k} : \mathcal{T}_B \to \mathcal{T}_A$$

is a contraction provided $\deg(f)^k > |A|$.

Proof. We may assume $|B| \geq 4$. Consider any 4-tuple $B_0 \subset B$. Since $|P(f)| = 3$, we have a point $b \in B_0 - V(f^k)$. Then $f^{-k}(b)$ is disjoint from $C(f^k)$, and $|f^{-k}(b)| = \deg(f)^k > |A|$, so we cannot have $f^{-k}(B_0) \subset A \cup C(f^k)$. ∎

Factorization. For later use, we record the following fact. Suppose we have a factorization $F = F_1 \circ F_2$. The pullback map can then be factored as

$$\mathcal{T}_B \xrightarrow{\sigma_{F_1}} \mathcal{T}_C \xrightarrow{\sigma_{F_2}} \mathcal{T}_A, \qquad (3.2)$$

where $C = F_2(A) \cup V(F_2)$.

Combinatorial equivalence. Finally we formulate the connection between fixed points on Teichmüller space and rational maps, following Thurston.

Let F and G be a pair of postcritically finite branched coverings of $\mathbb{P}^1$. An orientation–preserving homeomorphism of pairs

$$\phi : (\mathbb{P}^1, P(F)) \to (\mathbb{P}^1, P(G))$$

gives a *combinatorial equivalence* between F and G if there is a second homeomorphism ψ, isotopic to ϕ rel $P(F)$, making the diagram

$$\begin{array}{ccc} (\mathbb{P}^1, P(F)) & \xrightarrow{\psi} & (\mathbb{P}^1, P(G)) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ (\mathbb{P}^1, P(F)) & \xrightarrow{\phi} & (\mathbb{P}^1, P(G)) \end{array}$$

commute. Since $X = P(F)$ is forward invariant, F determines a holomorphic map

$$\sigma_F : \mathcal{T}_X \to \mathcal{T}_X.$$

The following result follows readily from the definitions (cf. [DH, Prop. 2.3]):

Proposition 3.3 *A point $[Y] \in \mathcal{T}_X$ is fixed by σ_F if and only if there exists a rational map f with $P(f) = Y$ such that the marking homeomorphism*

$$\phi : (\mathbb{P}^1, X) \to (\mathbb{P}^1, Y)$$

gives a combinatorial equivalence between F and f.

4 Prescribed dynamics on $P(f)$

In this section we prove Theorem 1.3. That is, given a finite set $X \subset \mathbb{P}^1$ with $|X| \geq 3$, and a map $F : X \to X$, we will construct a sequence of rigid rational maps f_n such that $P(f_n) \to X$ and

$$f_n|P(f_n) \to F|X.$$

This means that for all $n \gg 0$, we can find homeomorphisms ϕ_n of $\mathbb{P}^1$ such that $\phi_n \to \mathrm{id}$, $\phi_n(X) = P(f_n)$, and ϕ_n conjugates $F|X$ to $f_n|P(f_n)$.

The setup. Let h be a quadratic rational map with $J(h) = \mathbb{P}^1$ and

$$P(h) = \{0, 1, \infty\}.$$

(Explicitly, we can take $h(z) = (2/z - 1)^2$). Note that $V(h^n) = P(h)$ for all $n \geq 2$. The map h is expanding in the associated orbifold metric on $\mathbb{P}^1$ (see e.g. [Mil1, Thm 19.6], [Mc, App. A]).

It is convenient to normalize so that X contains $P(h)$. Let g be a polynomial fixing 0 and 1, such that its finite critical values $V_0(g)$ coincide with $X - P(h)$. (Such a polynomial exists by Lemma 2.2). Then

$$V(g \circ h^n) = V(g) \cup g(V(h^n)) = X$$

for all $n \geq 2$. For later convenience, we also choose g such that

$$\deg(g) = 3 \quad \text{if } |X| = 4. \tag{4.1}$$

(I.e. if $X = \{0, 1, \infty, a\}$, we take g to be a cubic polynomial with $V_0(g) = \{a\}$.)

Approximation by branched covers. Since $J(h) = \mathbb{P}^1$, the set $\bigcup_n h^{-n}(x)$ is dense for any $x \in \mathbb{P}^1$. Using this fact, we can construct a sequence of homeomorphisms

$$\phi_n : (\mathbb{P}^1, X) \to (\mathbb{P}^1, X_n),$$

with $\phi_n \to \mathrm{id}$, such that

$$F|X = g \circ h^n \circ \phi_n|X$$

for all n. To do this we first pick, for each $x \in X$, a nearby point x' such that $g \circ h^n(x') = F(x)$ and such that the map $x \mapsto x'$ is injective; set $X_n = \{x' : x \in X\}$. Then, we choose a homeomorphism ϕ_n close to the identity that moves x to x' for all $x \in X$. The larger n is, the closer we can take x' to x, and hence the closer we can take ϕ_n to the identity.

Construction of rational maps. Next we observe that

$$F_n = g \circ h^n \circ \phi_n : (\mathbb{P}^1, X) \to (\mathbb{P}^1, X)$$

is a smooth branched covering map with $P(F_n) = X$.

Theorem 1.3 follows from:

Theorem 4.1 *For all $n \gg 0$, F_n is combinatorially equivalent to a rational map f_n; and suitably normalized, we have $f_n|P(f_n) \to F|X$.*

Proof. Using the maps ϕ_n, we can regard X_n as points in $\mathcal{T}_X$ such that $X_n \to X$. By construction, we have

$$\sigma_{F_n}(X) = X_n,$$

and $d(X_n, X) \to 0$. We wish to control the contraction of σ_{F_n}, and produce a fixed point close to X. That is, to complete the proof it suffices to show there are point configurations $P_n \to X$ such that $\sigma_{F_n}(P_n) = P_n$. For then, by Proposition 3.3, we have a corresponding sequence of rational maps satisfy $P(f_n) = P_n$, and the marking homeomorphisms transport $F|X$ to $f_n|P(f_n)$.

Choose k such that $\deg(h)^k = 2^k > |X| + 3$. The crux of the matter is the factorization

$$F_n = (g \circ h^k) \circ (h^{n-k} \circ \phi_n),$$

valid for all $n \geq k$. From this we obtain a factorization of σ_{F_n} as

$$\mathcal{T}_X \xrightarrow{\sigma_g} \mathcal{T}_B \xrightarrow{\sigma_{h^k}} \mathcal{T}_A \xrightarrow{\sigma_{h^{n-k} \circ \phi_n}} \mathcal{T}_X;$$

see equation (3.2). In this factorization, we have

$$A = h^{n-k}(\phi_n(X)) \cup V(h^{n-k}).$$

Since $|V(h^{n-k})| \leq |P(h)| = 3$, we have $\deg(h)^k = 2^k > 3 + |X| \geq |A|$, and hence σ_{h^k} is a contraction by Proposition 3.2. Thus $\sigma_{g \circ h^k} = \sigma_{h^k} \circ \sigma_g$ is also a contraction. The amount of contraction at $P \in \mathcal{T}_X$ varies continuously with P. Since the ball of radius 2 about X in $\mathcal{T}_X$ is compact, we can find a constant λ, independent of n, such that

$$d(P, X) \leq 2 \implies \|D\sigma_{F_n}(P)\| \leq \|D\sigma_{g \circ h^k}(P)\| \leq \lambda < 1.$$

Let $X_n^i = \sigma_{F_n}^i(X)$, and let $\epsilon_n = d(X, X_n)$. For all $n \gg 0$, we have $\epsilon_n \ll (1 - \lambda)$. Under this assumption, we can prove by induction that:

(i) $d(X_n^i, X) \leq 1$, and hence
(ii) $d(X_n^i, X_n^{i-1}) \leq \epsilon_n \lambda^i$.

From this it follows that X_n^i converges, as $i \to \infty$, to a fixed point P_n of σ_{F_n} with $d(X, P_n) \leq \epsilon_n (1 - \lambda)^{-1}$. Since $\epsilon_n \to 0$, this completes the proof. ■

Proof of Theorem 1.3. We just need to verify that f_n is rigid. But if f_n is a flexible Lattès example, then $|P(f_n)| = |X| = 4$ and hence $\deg(g) = 3$ by condition (4.1). Moreover $\deg(f_n)$ is a square, contradicting the fact that $\deg(F_n) = \deg(g) \deg(h)^n = 3 \cdot 2^n$. ■

Algorithmic solution. We have used the construction above as the basis for a practical computer program that solves the approximation problem addressed by Theorem 1.3. The iteration does not quite take place on Teichmüller space; rather, we arrange that P_n is always close enough to X that there is a unique homeomorphism of $\mathbb{P}^1$ close to the identity sending P_n to X.

5 Solution to a Hurwitz problem

In this section we establish the existence of polynomials and rational maps with constrained branch data. This will enable us to strengthen Theorem 1.3 by prescribing local degrees at points of X as discussed in §6.

Our main result is:

Theorem 5.1 *Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a finite list of partitions. Then $\mathcal{P}$ can be extended to the passport of a polynomial if $n \geq 2$, and to the passport of a rational map if $n \geq 3$.*

Partitions. A *partition* P of $d \geq 0$ is a list of positive integers $(p_1, \ldots, p_s)$ such that $\sum p_i = d$. The *trivial partition* has $p_i = 1$ for $i = 1, \ldots, d$.

Given a second partition $P' = (p'_1, \ldots, p'_t)$ of $d' = \sum p'_i$, we let

$$P + P' = (p_1, \ldots, p_s, p'_1, \ldots, p'_t) \tag{5.1}$$

denote the combined partition of $d + d'$.

Passports. A *passport* of degree d is a finite list $\mathcal{P} = (P_1, \ldots, P_n)$ of nontrivial partitions of d. For both partitions and passports, repetitions are allowed and the order in which elements appear is unimportant. We set

$$c(\mathcal{P}) = \sum_i (d - |P_i|).$$

Rational maps. For any y in the target of a rational map f we have a partition $P(f, y)$ of $d = \deg(f)$ given by:

$$\sum_{f(x)=y} \operatorname{mult}(f, x).$$

The *passport of f* is the collection of partitions

$$\mathcal{P}(f) = (P(f, v_1), \ldots, P(f, v_n))$$

arising from the critical values $\{v_1, \ldots, v_n\}$ of f. (The other points in the target of f yield trivial partitions.) The number of critical points of f mapping to a given point y, counted with multiplicity, is $d - |P(f, y)|$. Hence

$$c(\mathcal{P}(f)) = 2d - 2.$$

Branched coverings. The passport of a smooth branched covering map $F : S^2 \to S^2$ is defined similarly. As is well known, any passport that can be realized topologically can be realized geometrically. More precisely, we have:

Proposition 5.2 *Let $F : S^2 \to S^2$ be a branched covering with $V(F) = \{v_1, \ldots, v_n\}$, and let $X = \{x_1, \ldots, x_n\} \subset \mathbb{P}^1$. Then there exists a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ with $\deg(f) = \deg(F)$ and $V(f) = X$ such that*

$$P(f, x_i) = P(F, v_i) \ \textit{for } i = 1, \ldots, n.$$

In particular, $\mathcal{P}(f) = \mathcal{P}(F)$.

Proof. Choose an orientation–preserving diffeomorphism $\phi : S^2 \to \mathbb{P}^1$ such that $\phi(v_i) = x_i$ for $i = 1, \ldots, n$. Pulling the complex structure on $\mathbb{P}^1$ back to S^2 via $\phi \circ F$, and applying the uniformization theorem, we obtain a homeomorphism $\psi : S^2 \to \mathbb{P}^1$ such that $f = \phi \circ F \circ \psi^{-1}$ is a holomorphic branched covering, and hence a rational map (cf. [Thom]). ■

Hurwitz problem. The *Hurwitz problem* is to characterize the passports that arise from branched coverings of S^2. A complete solution is not known; for background, see e.g. [LZ, Ch. 5]. Theorem 5.1 addresses a variant of this problem where we allow the partitions to be *extended*.

Extensions. A partition Q *extends* P if $Q = P + P'$ for some partition P'. (For example, $1 + 3 + 5 + 7 = 16$ is an extension of $3 + 7 = 10$.)

Our main interest is in finite collections of partitions $\mathcal{P} = (P_1, \ldots, P_n)$, with repetitions allowed. In this setting we say that $\mathcal{Q}$ *extends* $\mathcal{P}$ if, when suitably ordered, we have $\mathcal{Q} = (Q_1, \ldots, Q_n)$ and Q_i extends P_i for $i = 1, \ldots, n$. In particular, if $\mathcal{Q}$ extends $\mathcal{P}$ then $|\mathcal{Q}| = |\mathcal{P}|$.

Polynomials. When $g : \mathbb{C} \to \mathbb{C}$ is a *polynomial*, its passport $\mathcal{P}(g)$ is defined as the list of partitions $P(g, v_i)$ coming from the *finite* critical values of g.

The passports of polynomials are easily described. In fact, by [EKS, Prop. 5.2] we have:

Theorem 5.3 *A passport $\mathcal{P} = (P_1, \ldots, P_n)$ of degree d arises from a polynomial g if and only if*

$$c(\mathcal{P}) = \sum_i (d - |P_i|) = d - 1. \tag{5.2}$$

The equation above is necessary because g has $d-1$ critical points. Applying Proposition 5.2, we obtain:

Corollary 5.4 *Let $X \subset \mathbb{C}$ be a finite set such that $1 \le |X| < d$. Then there exists a polynomial g of degree d whose critical values coincide with X.*

Corollary 5.5 *Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a collection of partitions with $n \ge 2$. Then $\mathcal{P}$ can be extended to the passport of a polynomial of degree d for all d sufficiently large.*

Proof. It will be convenient to use exponential notation for repeated integers (so (1^d) is a partition of d).

First, extend the partitions in $\mathcal{P}$ so they are all nontrivial partitions of the same integer d. Then $\mathcal{P}$ is a passport. If we extend P_i to $P_i + (1)$ for all i, then d increases by 1 but $c(\mathcal{P})$ remains the same. Thus after a further extension of $\mathcal{P}$, we can assume that $d-1 \geq c(\mathcal{P})$. If equality holds, we are done.

Otherwise, extend P_i to $P_i+(3)$ for $i=1,2$, and to $P_i+(1,1,1)$ for $i \geq 3$. Then d increases by 3 while $c(\mathcal{P})$ increases by 4. By repeating this type of extension until equality holds in equation (5.2), we obtain an extension of $\mathcal{P}$ that arises from a polynomial of degree d.

Finally, suppose $\mathcal{P}$ is the passport of a polynomial of degree d. To complete the proof, we will show that $\mathcal{P}$ can be extended to a polynomial passport of degree $d+k$ for any $k \geq 2$. To see this, just extend P_1 to $P_1+(k)$, P_2 to $P_2+(2,1^{k-2})$, and P_i to $P_i+(1^k)$ for $i \geq 3$. ∎

Theorem 5.6 *Any collection of partitions $\mathcal{P}=(P_1,\ldots,P_n)$ with $n \geq 3$ can be extended to the passport of a rational map.*

Proof. We divide the proof into two cases.

Case I. Assume $n \geq 4$. Let $\mathcal{P}_1=(P_1,P_2)$ and let $\mathcal{P}_2=(P_3,\ldots,P_n)$. By Corollary 5.5, we may assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ are passports of polynomials g_1 and g_2 of the same degree d.

The complex plane can be naturally completed to a closed disk

$$D \cong \mathbb{C} \cup S^1$$

by adding a circle to represent the rays in $T_\infty \widehat{\mathbb{C}}$. Then each polynomial g_i extends continuously to a proper map D_i of degree d on D, satisfying $D_i(x)=dx$ for all $x \in S^1 \cong \mathbb{R}/\mathbb{Z}$.

Now construct a branched covering $F: S^2 \to S^2$ by gluing together two copies of D to obtain a sphere, and then setting $F=D_1$ on the first copy and $F=D_2$ on the second. Then $\mathcal{P}(F)=\mathcal{P}$ by construction, and F can be replaced by a rational map by Proposition 5.2, and the proof in this case is complete.

Case II. Now assume that $n=|\mathcal{P}|=3$. We will construct a rational map with $V(f)=\{0,1,\infty\}$ such that $\mathcal{P}(f)$ extends $\mathcal{P}$.

For convenience, let us index the elements P_i of $\mathcal{P}$ by $i \in I=\{0,1,\infty\}$. After replacing P_i by the extension $P_i+(2)$, if necessary, we may assume that every partition in $\mathcal{P}$ is nontrivial.

To construct f, it suffices to give the topological data of a *dessin d'enfant* $D \subset \mathbb{C}$ [Sn], [LZ]. A dessin is connected graph with vertices of two colors, embedded in the plane, arising as the preimage of the interval $[0,1]$ under

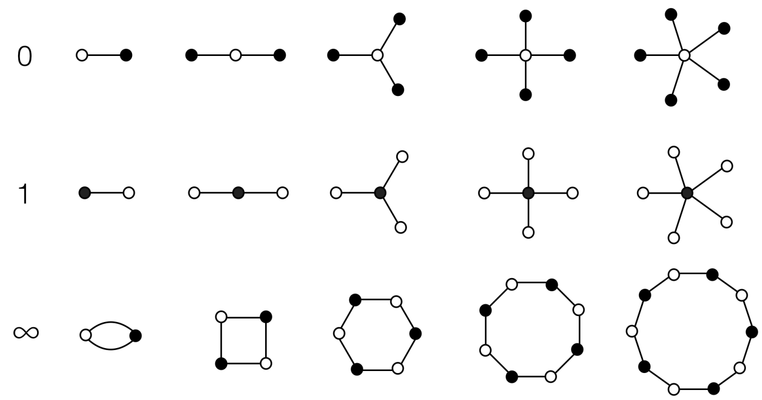

Figure 1: Model dessins $D_0(m), D_1(m)$, and $D_\infty(m)$ are drawn in rows for $1 \leq m \leq 5$.

a branched covering $F : \mathbb{P}^1 \to \mathbb{P}^1$ with $V(F) \subset \{0, 1, \infty\}$. We adopt the convention that the white vertices of D map to 0 and the black vertices map to 1. The components of $\mathbb{P}^1 - D$ are called the *faces* of D, and each face contains a unique point z such that $F(z) = \infty$.

Consider, for each $m \geq 1$, the three types of model dessins

$$D_0(m), \quad D_1(m), \quad \text{and} \quad D_\infty(m)$$

shown in Figure 1. (These graphs correspond to the rational maps $f_0(z) = z^m$, $f_1(z) = z^m + 1$ and $f_\infty(z) = (z^m + z^{-m} + 2)/4$.) The barycenter z of $D_i(m)$ is a vertex for $i = 0$ or $i = 1$, and it lies in a face if $i = \infty$. In all three cases, $F(z) = i$ and $\text{mult}(F, z) = m$.

Let $P_0 = (a_1, \ldots, a_n)$, and let $G_0 \subset \mathbb{C}$ be a planar graph with n components, such that its jth component is isomorphic to $D_0(a_j)$. Such a graph is unique up to planar isotopy. Construct $G_1 \subset \mathbb{C}$ and $G_\infty \subset \mathbb{C}$ similarly, using the partitions P_1, P_∞ and the models $D_1(m), D_\infty(m)$. When constructing G_∞, we take care not to nest two components of type $D_\infty(m)$. It is easy to arrange that the graphs G_i are contained in disjoint disks in the plane. Let $G = G_0 \cup G_1 \cup G_2$. Then every vertex of G is incident to the unique unbounded component U of $\mathbb{C} - G$.

To obtain D, we add new edges between vertices of opposite colors to make G connected. In the process we take care not to alter the valence at the center of each component of G. This can be done in many ways.

For example, begin by introducing a new edge $[b, w] \subset U$, with one white vertex w and one black vertex b. Then connect each component of G_0 to w,

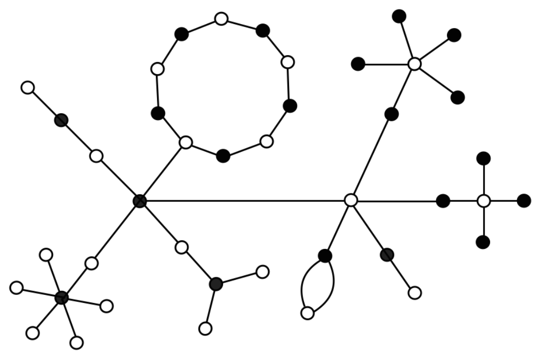

Figure 2: A dessin d'enfant D obtained by connecting the components of G to a single edge $[b, w]$, which is drawn in the center. All four components of G_0 are connected to w; all four components of G_1 are connected to b; and one component of G_∞ is connected to w while the other is connected to b. The corresponding rational map f has degree 42.

connect each component of G_1 to b, and connect each component of G_∞ to either w or b as in Figure 2, making sure that these new edges do not cross.

The resulting connected graph $D \subset \mathbb{C}$ is then a dessin d'enfant for a rational map f with $V(f) = \{0, 1, \infty\}$. By the construction of the graph G_i, the partition $\mathcal{P}(f, i)$ extends P_i for each $i \in V(f)$, and hence $\mathcal{P}(f)$ extends $\mathcal{P}$. ∎

Proof of Theorem 5.1. The statements for polynomials and for rational maps are covered by Corollary 5.5 and Theorem 5.6 respectively. ∎

Remarks and references. The proof of Theorem 5.6 for $n \geq 4$ is based on the fact that the passports for a pair of polynomials of degree d can be combined to give the passport of a rational map. This result also appears in [EKS, Remark on p. 785] and [Bar, Prop. 10]. The branched covering built from a pair of polynomials is called their *formal mating* in [Tan].

6 Prescribed critical points in $P(f)$

In this section we strengthen Theorem 1.3 by showing that we can construct a rational map with prescribed critical points in $P(f) \approx X$. We will also give a similar result for polynomials.

Multiplicities. To make the first statement precise, recall that f has a critical point of order $\mathrm{mult}(f,x)-1$ at each point $x \in \mathbb{P}^1$. We can regard the multiplicity as a map $\mathrm{mult}(f) : \mathbb{P}^1 \to \mathbb{Z}_+ = \{1,2,3,\ldots\}$.

Our aim is to show:

Theorem 6.1 *Let $F : X \to X$ and $M : X \to \mathbb{Z}_+$ be arbitrary maps defined on a finite set $X \subset \mathbb{P}^1$ with $|X| \geq 3$. Then there exists a sequence of rigid, postcritically finite rational maps f_n such that $|P(f_n)| = |X|$,*

$$P(f_n) \to X, \quad f_n|P(f_n) \to F|X, \quad \textit{and} \quad \mathrm{mult}(f_n)|P(f_n) \to M|X$$

as $n \to \infty$.

Proof. The argument is a modification of the proof of Theorem 1.3 given in §4.

As in that section, the construction is based on a pair of rational maps g and h. Let h be a quadratic rational map with $J(h) = \mathbb{P}^1$ and $|P(h)| = 3$.

To construct g, we first associate to each $x \in X$ the partition

$$P_x = (M(y_1), \ldots, M(y_n))$$

where $F^{-1}(x) = \{y_1, \ldots, y_n\} \subset X$. Then $\sum_X |P_x| = |X|$. (Note that if $x \notin F(X)$, then P_x is the empty partition and $|P_x| = 0$.)

Let $P'_x = P_x + (1)$ be the partition obtained by padding P_x with an extra 1 at the end (notation as in equation (5.1)), so $|P'_x| = |P_x| + 1$. By Proposition 5.2 and Theorem 5.6, there is a rational map g with critical values $V(g) = X$ such that $P(g,x)$ extends P'_x for all $x \in X$. Since $|X| \geq 3$, we then have

$$|g^{-1}(X)| = \sum_X |P(g,x)| \geq \sum_X |P'_x| = 2|X| \geq |X| + 3. \tag{6.1}$$

Next, we construct an injective map

$$\iota : X \to g^{-1}(X) \subset \mathbb{P}^1$$

such that

$$F(y) = g(\iota(y)) \quad \text{and} \quad M(y) = \mathrm{mult}(g, \iota(y)) \tag{6.2}$$

for all $y \in X$. To define $\iota(y)$, let $x = F(y)$ and recall that: y determines a point $M(y) \in \mathcal{P}_x$, we have an inclusion $\mathcal{P}_x \subset P(g,x)$, and there is a bijection $P(g,x) \cong g^{-1}(x)$ which labels points by their multiplicities. We define $\iota(y)$ to be the image of $M(y)$ under the composition $\mathcal{P}_x \subset P(g,x) \cong g^{-1}(x)$.

Since $|P(h)| = 3$ and equation (6.1) holds, we can choose $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\alpha(P(h)) \subset g^{-1}(X) - \iota(X)$. Upon replacing g and ι with $g \circ \alpha$ and $\alpha^{-1} \circ \iota$, equation (6.2) continues to hold, and we then have

$$P(h) \subset g^{-1}(X) - \iota(X).$$

Now recall that $J(h) = \mathbb{P}^1$ and hence

$$X \subset J(h). \tag{6.3}$$

Consequently, for any $x \in X$, the inverse orbit of $\iota(x)$ under h accumulates on every point of X. Moreover, if $h^n(z) = \iota(x)$, then $\mathrm{mult}(h^n, z) = 1$, since $\iota(x) \notin P(h)$. Thus we can find a sequence of injective maps $\phi_n : X \to X_n$, converging to the identity, such that

$$\mathrm{mult}(h^n)|X_n = 1 \quad \text{and} \quad h^n(\phi_n(x)) = \iota(x). \tag{6.4}$$

Extend $\langle \phi_n \rangle$ to a sequence of homeomorphisms of $\mathbb{P}^1$ converging to the identity as $n \to \infty$, and let

$$F_n = g \circ h^n \circ \phi_n : (\mathbb{P}^1, X) \longrightarrow (\mathbb{P}^1, X). \tag{6.5}$$

Then by equation (6.2), we have

$$F_n|X = F \quad \text{and} \quad \mathrm{mult}(F_n)|X = M|X. \tag{6.6}$$

Finally, we reiterate the proof of Theorem 4.1 to convert the postcritically finite branched covers F_n into rational maps f_n with $P(f_n) \to X$. Since f_n and F_n are conjugate on their postcritical sets, we then have

$$f_n|P(f_n) \to F|X \quad \text{and} \quad \mathrm{mult}(f_n)|P(f_n) \to M|X$$

by equation (6.6). We can also ensure, by our choice of g, that $\deg(f_n)$ is not a square, and hence the postcritically finite maps f_n are rigid for all $n \gg 0$. ■

The polynomial case. We conclude by presenting a variation of Theorem 6.1 for polynomials.

Theorem 6.2 *Let $F : X \to X$ and $M : X \to \mathbb{Z}_+$ be arbitrary maps defined on a finite set $X \subset \mathbb{C}$ with $|X| \geq 2$. Then there exists a sequence of postcritically finite polynomials f_n such that $|P_0(f_n)| = |X|$,*

$$P_0(f_n) \to X, \quad f_n|P_0(f_n) \to F|X, \quad \text{and} \quad \mathrm{mult}(f_n)|P_0(f_n) \to M|X$$

as $n \to \infty$.

Prescribed Julia sets. For the proof, we will need a polynomial h whose Julia set contains X (to play the role of the rational map h with $J(h) = \mathbb{P}^1$ in the proof of Theorem 6.1). It suffices to treat the case where $X \subset \overline{\mathbb{Q}}$, since $\overline{\mathbb{Q}}$ is dense in $\mathbb{C}$.

Lemma 6.3 *Given any finite set $X \subset \overline{\mathbb{Q}}$, there is a polynomial h so that*

$$|P_0(h)| = 2 \quad \textit{and} \quad X \subset J(h).$$

Proof. By Theorem 2.3, there is a polynomial β with $\beta(X) \cup V_0(\beta) \subset \{0, 1\}$. We can assume that $\deg(\beta) > 1$ and $V_0(\beta) = \{0, 1\}$ (for example, by taking a Belyi polynomial for a larger set that contains X).

There are two distinct points $a, b \in \beta^{-1}(\{0, 1\})$ that are not critical points of β. Indeed, the set $\beta^{-1}(\{0, 1\})$ consists of $2d$ points, counted with multiplicity, and at most $2|C_0(f)| = 2d - 2$ of these are accounted for by critical points.

Let $\alpha \in \mathrm{Aut}(\mathbb{C})$ be an affine transformation sending the ordered pair (a, b) to $(0, 1)$, and set $h = \alpha \circ \beta$. We then have:

$$V_0(h) = \{a, b\}, \quad h(\{a, b\}) \subset \{a, b\}, \quad \text{and} \quad h(X) \subset \{a, b\}.$$

The first two properties imply that $P_0(h) = \{a, b\}$, and in particular $P_0(h)$ is disjoint from $C_0(h)$. It follows that $P_0(h)$ is contained in the Julia set of h. Since the Julia set is totally invariant and $h(X) \subset \{a, b\}$, we have $X \subset J(h)$ as well. ∎

Proof of Theorem 6.2. We may assume $X \subset \overline{\mathbb{Q}}$. Let h be a polynomial associated to X as in Lemma 6.3. Let $(P'_x : x \in X)$ be the family of partitions constructed in the proof of Theorem 6.1, and let g be a polynomial with $V_0(g) = X$, provided by Proposition 5.2 and Corollary 5.5, such that $P(g, x)$ extends P'_x for all $x \in X$. We can now simply repeat the proof of Theorem 6.1, using the mappings g and h to obtain the desired polynomials f_n. ∎

7 Alternative constructions

In this section we discuss alternative constructions of rational maps with postcritical sets satisfying $|P(f)| \leq 4$.

The case $|P(f)| \leq 3$. The only rational maps with $|P(f)| = 2$ are those that are conformally conjugate to $z \mapsto z^{\pm d}$. Question 1.2 then has a negative answer if $X \subset \mathbb{P}^1$ has cardinality 2; that is, $F : X \to X$ is realized by $f : P(f) \to P(f)$ if and only if F is bijective.

It is easy to see that Question 1.2 has a positive answer whenever $|X| = 3$. Indeed, we can first normalize so that $X = \{0, 1, \infty\}$. Then, up to reordering the points of X, there are only seven possibilities for F. A concrete rational map realizing each one is given in Figure 3.

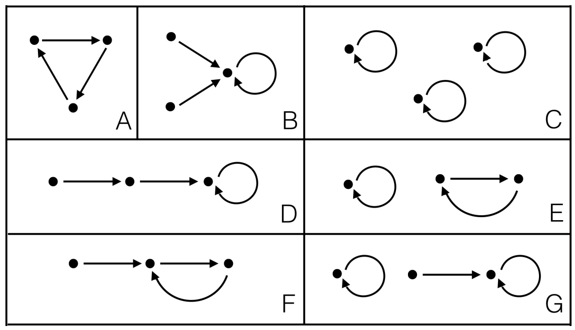

Figure 3: This table contains all seven possible graphs representing maps $X \to X$ when $|X| = 3$. Each graph can be realized by a rational map $f : P(f) \to P(f)$. Explicit examples are given by $A(z) = 1 - 1/z^2$; $B(z) = (z - \alpha)^3/(z - 1 + \alpha)^3$ where $\alpha^2 - \alpha + 1 = 0$; $C(z) = z^2(3 - 2z)$; $D(z) = (1 - 2/z)^2$; $E(z) = z^2 - 1$; and $F(z) = (2z-1)^2/(4z(z-1))$; and $G(z) = z^2 - 2$. In each case, the degree is minimal among all rational maps realizing the given dynamics on X.

The case $|P(f)| = 4$: rigid Lattès maps. Let $Q \subset \mathbb{H}$ denote the set of τ in the upper halfplane that are quadratic over $\mathbb{Q}$, and let

$$\lambda : \mathbb{H} \to \mathbb{H}/\Gamma(2) \cong \mathbb{P}^1 - \{0, 1, \infty\}$$

be the universal covering map. Rigid Lattès maps are precisely those which are covered by the action of complex multiplication on an elliptic curve E (up to translation by a point of order in $E[2]$); see [Mil2, Lemmas 4.3 and 4.4]. Every such elliptic curve has the form $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with $\tau \in Q$. Using these rigid maps, one can explicitly construct f with $P(f) = \{0, 1, \infty, z\}$ for all z in the dense set $\lambda(Q) \subset \mathbb{P}^1$. On the other hand,

$$L = \lambda(Q) \tag{7.1}$$

is very small subset of $\overline{\mathbb{Q}}$. For example, $L \cap \mathbb{Z} = \{-1, 2\}$, since the j–invariant

$$j = \frac{256(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2}$$

is an algebraic integer for all $\lambda \in L$ (see e.g. [Sil, §II.6] for details). So the postcritical sets arising from rigid Lattès examples are insufficient to complete the proof of Theorem 1.1 in the case $|P(f)| = 4$.

The case $|P(f)| = 4$: dynamics on moduli space. Our second construction of rational maps with given postcritical sets is based on [Ko]. There, the author builds maps on moduli space $g : \mathcal{M}_{0,n} \dashrightarrow \mathcal{M}_{0,n}$ whose periodic points correspond to postcritically finite rational maps on the Riemann sphere.

For $n = 4$, the moduli space $\mathcal{M}_{0,4}$ can be identified with $\mathbb{P}^1 - \{0, 1, \infty\}$, and $g(z) = (1 - 2/z)^2$ is an example of one such map on moduli space. Each periodic point x of period m gives rise to a postcritically finite polynomial of degree 2^m with postcritical set $\{0, 1, \infty, x\}$. Let K denote the set of all periodic points of g in $\mathbb{P}^1 - \{0, 1, \infty\}$. The set K lies in $\overline{\mathbb{Q}}$, and it is dense in the Julia set $J(g)$. Moreover, we have $J(g) = \mathbb{P}^1$ since g is a Lattès map, so $K \subset \mathbb{P}^1$ provides a dense set of postcritical sets arising from polynomials.

Interestingly, the intersection $L \cap K$ is finite (where L is defined by equation (7.1)). Indeed, K is a set of bounded Weil height, as a consequence of the existence of a canonical height for g [CS]. But the j-invariants of elliptic curves with complex multiplication have no infinite subsets of bounded height [Po, Lemma 3], so neither does L.

References

[Bar] K. Barański. On realizability of branched coverings of the sphere. *Topol. Appl.* **116**(2001), 279–291.

[Bel] G. V. Belyi. On Galois extensions of a maximal cyclotomic field. *Math. USSR Izv.* **14**(1980), 247–256.

[BCT] X. Buff, G. Cui, and Tan L. Teichmüller spaces and holomorphic dynamics. In *Handbook of Teichmüller theory. Vol. IV*, pages 717–756. Eur. Math. Soc., 2014.

[CS] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.* **89**(1993), 163–205.

[DH] A. Douady and J. Hubbard. A proof of Thurston's topological characterization of rational maps. *Acta Math.* **171**(1993), 263–297.

[EKS] A. Edmonds, R. Kulkarni, and R. E. Stong. Realizability of branched coverings of surfaces. *Trans. Amer. Math. Soc.* **282**(1984), 773–790.

[Hub] J. H. Hubbard. *Teichmüller Theory*, volume II. Matrix Editions, 2016.

[Ko] S. Koch. Teichmüller theory and critically finite endomorphisms. *Adv. Math.* **248**(2013), 573–617.

[LZ] S. K. Lando and A. K. Zvonkin. *Graphs on Surfaces and Their Applications*. Springer, 2004.

[Mc] C. McMullen. *Complex Dynamics and Renormalization*, volume 135 of *Annals of Math. Studies.* Princeton University Press, 1994.

[Mil1] J. Milnor. *Dynamics in One Complex Variable*, volume 160 of *Annals of Math. Studies.* Princeton University Press, 2006.

[Mil2] J. Milnor. On Lattès maps. In P. G. Hjorth and C. L. Petersen, editors, *Dynamics on the Riemann sphere*, pages 9–44. European Math. Soc., 2006.

[Po] B. Poonen. Spans of Hecke points on modular curves. *Math. Res. Lett.* **8**(2001), 767–770.

[Sn] L. Schneps, editor. *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, 1994.

[Sil] J. Silverman. *Advanced topics in the Arithmetic of Elliptic Curves.* Springer–Verlag, 1994.

[Tan] Tan L. Matings of quadratic polynomials. *Ergod. Th. & Dynam. Sys.* **12**(1992), 589–620.

[Thom] R. Thom. L'equivalence d'une fonction différentiable et d'un polynôme. *Topology* **3**(1965), 297–307.