PERIODIC ORBITS AND KNEADING INVARIANTS

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Let I be a closed real interval containing 0 in its interior, and let $f: I \to I$ be a continuous function which has either one maximum or one minimum at 0, and is strictly monotone on each component of $I \backslash \{0\}$. Such a function stretches and then folds the interval into itself; that is, it 'kneads' the interval. We shall also assume that $f(\partial I) \subset \partial I$. For a specified point $x \in I$ we can study the orbit $\{f^n(x)\}_{n=0,1,2,\ldots}$. In particular, we shall be interested in the existence of periodic orbits.

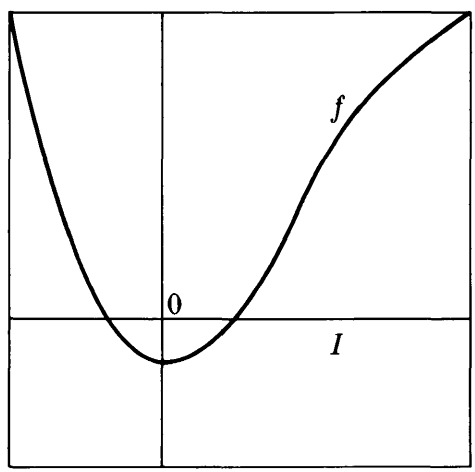

FIG. 1

This question of the existence and the number of periodic orbits is of interest whenever there is an iterative process whose mathematical formulation takes the above form. Much of the structure of the function is determined by its periodic orbits. For example, in a model for population biology, $f(x)$ could represent the size of a population of a certain species given that one generation ago it was measured at x. This paper does not propose a model that could be used by a population biologist. Rather, it suggests how very complex the behaviour can be even when a very simple iterative model is used. At the same time it will appear that this complex behaviour evolves under perturbation of the function f with a certain mathematical order of its own.

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The first theorems deal with the existence of periodic orbits. Both x and the orbit of x are said to be *periodic* if $f^n(x) = x$ for some integer n. The smallest such n is called the *minimal period* of the orbit and of the point.

Let $\mathbf{Z}_+$ denote the set of all integers greater than 0. We define an ordering $\lhd$ on $\mathbf{Z}_+$ as follows. If k, l are odd and $k, l \geqslant 3$, then $2^n k \lhd 2^n l$ if $k < l$. If k, l are odd and $k, l \geqslant 3$, and $m > n$, then $2^n k \lhd 2^m l$. If k is odd and $k \geqslant 3$ then $2^n k \lhd 2^m$. Finally, $2^m \lhd 2^n$ if $n < m$. Thus, we have

$$3 \lhd 5 \lhd 7 \lhd 9 \lhd \ldots \lhd 2.3 \lhd 2.5 \lhd 2.7 \lhd \ldots \lhd 4.3 \lhd \ldots \lhd 8 \lhd 4 \lhd 2 \lhd 1.$$

The first theorem is as follows:

THEOREM A. *Let $f: I \to I$ be a continuous function with precisely one maximum or minimum, located at* 0, *and suppose $f(\partial I) \subset \partial I$. Then, if f has a periodic orbit of minimal period n, it has a periodic orbit of minimal period m for every m such that $n \lhd m$.*

This theorem is not new. In 1964 Šarkovskii [**5**] proved the theorem for any continuous function $f: \mathbf{R} \to \mathbf{R}$. That is, if f has a periodic orbit of minimal period n, then f has a periodic orbit of minimal period m for every integer m such that $n \lhd m$. A recent English account of Šarkovskii's proof is given by Štefan in [**6**]. A paper by Li and Yorke [**3**] treats a special case. Li and Yorke's paper gives a good indication of the method of proof used by Šarkovskii. It is quite different from the procedure followed below. Independently, Guckenheimer recently also obtained Theorem A assuming f differentiable [**2**].

The method we use to prove this theorem and the other theorems derives from an idea of Thurston and Milnor (see [**4**]) who associate to f and also to each point of I formal power series called, respectively, the *kneading invariant* and the *invariant coordinates*. These power series contain information about the function. In particular, if a point is periodic, its invariant coordinate is periodic. All the information required to determine the existence of periodic orbits of every order is contained in the kneading invariant of f. In fact, it gives a lower bound on the number of periodic orbits of each order. For example,

THEOREM B. *Let n and l be odd numbers such that $n > l \geqslant 3$. Suppose f has an orbit of minimal period l. Then f has at least $2^{(n-l)/2}$ distinct orbits of minimal period n.*

If f is assumed to be differentiable, the method described in this paper also allows one to determine precisely how bifurcations give rise to new periodic orbits when the function f is perturbed.

Theorems C and D indicate that there is a sudden jump in the complexity of the behaviour of f when the kneading invariant of f reaches a certain value λ defined below. When it is greater than λ there are essentially a finite number of periodic orbits, each of minimal period 2^n for some n, and all orbits are asymptotic to one of these. As soon as the kneading invariant of f reaches λ, there will be infinitely many orbits that are not asymptotic to periodic orbits.

The precise formulation of Theorems C and D is given below.

The paper is divided as follows. In § 1 we give an exposition of the relevant parts of Milnor and Thurston's theory of kneading. This material has appeared only in handwritten form in a set of notes which is not readily available, so it is necessary to include some of the details here. Essentially all of § 1 is taken from [4], as is Lemma 4.0. In § 2 we find the precise relationship between periodic orbits and periodic kneading invariants as expressed in the main theorem of this section (Theorem 2.4). In § 3 we find the maximal periodic or anti-periodic kneading invariant for each period. The result is contained in Theorem 3.4 and applied in Theorem 3.5 and the proof of Theorem B. In § 4 we study the relationship between the asymptotic behaviour of a point x and the eventually periodic form of its invariant coordinate. This culminates in Theorems C and D.

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1. The theory of kneading

Given a point $x \in I$ we can describe very roughly the orbit of the point x by indicating for each $n \in \mathbf{Z}_+$ whether $f^n(x)$ is to the left or to the right of 0. For convenience we assume that f has a minimum at 0. The case of a function with one maximum can always be reduced to this one by considering $\sigma \circ f \circ \sigma$ where σ is a homeomorphism of I that reverses orientation.

We let

$$\varepsilon_n(x) = \begin{cases} 0 & \text{if } f^n(x) = 0, \\ 1 & \text{if } f^n(x) > 0, \\ -1 & \text{if } f^n(x) < 0. \end{cases}$$

We express this information in the formal power series $\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon_i(x)t^i$. Essentially we are partitioning I by the sets $f^{-n}(0)$ for each n. The numbers $\varepsilon_0(x), \varepsilon_1(x), \ldots$ indicate which of the resulting intervals contains x. Let $\mathbf{Z}[[t]]$ denote the set of formal power series with integral coefficients.

Unfortunately the mapping $x \to \varepsilon(x)$ of I into $\mathbf{Z}[[t]]$ is not order-preserving in any reasonable sense, precisely because f reverses orientation on one component of $I \setminus \{0\}$. To correct this we associate with x a second formal power series called the *invariant coordinate* $\theta(x)$ of x. We do this by composing the mapping $x \to \varepsilon(x)$ with a mapping θ of $\mathbf{Z}[[t]]$ into itself. If $\varepsilon = \sum_{i=0}^{\infty} \varepsilon_i t^i \in \mathbf{Z}[[t]]$ we define

$$\theta(\varepsilon) = \sum_{n=0}^{\infty} \theta_n(\varepsilon) t^n$$

where $\theta_n(\varepsilon) = \prod_{i=0}^{n} \varepsilon_i$. We will usually write $\theta(x)$ for $\theta(\varepsilon(x))$ and $\theta_n(x)$ for $\theta_n(\varepsilon(x))$. Note that the set of polynomials with integral coefficients can be identified with the subset of $\mathbf{Z}[[t]]$ consisting of formal power series with only a finite number of non-vanishing coefficients. It follows from the definition of $\theta(x)$ that all coefficients of $\theta(x)$ equal 1, -1, or 0. Moreover, if $\theta_n(x) = 0$ then $\theta_m(x) = 0$ for all $m \geqslant n$, and $\theta(x)$ is a polynomial of degree n if and only if n is the smallest integer such that $f^{n+1}(x) = 0$. In particular, if 0 is not contained in the orbit of x, then $\theta(x)$ has no vanishing coefficients, and conversely. Note that if $\theta_{i-1}(x) \neq 0$ then

$$\varepsilon_i(x) = \theta_i(x)/\theta_{i-1}(x).$$

The following formula will prove useful. If none of $x, f(x), \ldots, f^k(x)$ vanishes, then

$$\begin{aligned}\theta_i(f^k(x)) &= \varepsilon_k(x)\varepsilon_{k+1}(x)\ldots\varepsilon_{k+i}(x) \\ &= \theta_{k+i}(x)/\theta_{k-1}(x) \\ &= \theta_{k-1}(x)\theta_{k+i}(x).\end{aligned}$$

That is,

$$\theta(f^k(x)) = \theta_{k-1}(x)\{\theta_k(x) + \theta_{k+1}(x)t + \theta_{k+2}(x)t^2 + \ldots\}.$$

We have that f is monotone increasing at y if and only if $\varepsilon_0(y) = 1$, and f is monotone decreasing at y if and only if $\varepsilon_0(y) = -1$.

If $x, y \in I$ are distinct, let $C(x,y)$ be the closed interval with x and y as end points.

LEMMA 1.0. *If $\varepsilon(x) = \varepsilon(y) \pmod{t^N}$ then $f^k | C(x,y)$ is a homeomorphism onto its image for every $k \leqslant N$, and $f^k(x), f^k(y) \neq 0$ for all $k < N$. Conversely, if $f^k | C(x,y)$ is a homeomorphism for every $k \leqslant N$, and if $f^k(x), f^k(y) \neq 0$ for all $k < N$, then $\varepsilon(x) = \varepsilon(y) \pmod{t^N}$.*

Proof. The proof is left to the reader.

Now suppose $x < y$ and let n be the first non-negative integer for which $\varepsilon_n(x) \neq \varepsilon_n(y)$. Thus $\varepsilon(x) = \varepsilon(y) \pmod{t^n}$. By Lemma 1.0, $f^n | [x,y]$ is a homeomorphism and $\varepsilon_k(x) \neq 0$ for $k < n$. The homeomorphism $f^n | [x,y]$

will be sense-preserving (respectively sense-reversing) if and only if

$$\theta_{n-1}(x) = \varepsilon_0(x)\varepsilon_1(x)\varepsilon_2(x)\ldots\varepsilon_{n-1}(x) > 0 \quad \text{(respectively } < 0).$$

Thus if $\theta_{n-1}(x) > 0$ (respectively < 0) we will have $\varepsilon_n(x) < \varepsilon_n(y)$ (respectively $\varepsilon_n(x) > \varepsilon_n(y)$). In either case,

$$\theta_n(x) = \theta_{n-1}(x)\varepsilon_n(x) < \theta_{n-1}(x)\varepsilon_n(y) = \theta_{n-1}(y)\varepsilon_n(y) = \theta_n(y).$$

Therefore, if we endow $\mathbf{Z}[[t]]$ with the usual lexicographical ordering we get the following proposition:

PROPOSITION 1.1 (Milnor [4]). *The function $\theta: I \to \mathbf{Z}[[t]]$ is monotone increasing.*

Milnor also defines the *kneading invariant*, an invariant of the function, as follows:

$$\nu(f) = \lim_{x\downarrow 0}\theta(x).$$

Here and throughout the paper, we use as topology on $\mathbf{Z}[[t]]$ the one induced by the metric

$$\rho(\varphi,\alpha) = \sum_{i=0}^{\infty}\frac{|\varphi_i-\alpha_i|}{2^{i+1}}.$$

We define numbers ν_i by writing $\nu(f) = \sum_{i=0}^{\infty}\nu_i t^i$. Since for every value of n the set $f^{-n}(0)$ is finite, each of these coefficients is different from zero. Furthermore, $\nu_0 = 1$.

We let $\theta(x+)$ denote $\lim_{y\downarrow x}\theta(y)$ and we let $\theta(x-)$ denote $\lim_{y\uparrow x}\theta(y)$.

PROPOSITION 1.2 (Milnor [4]). *If $f^n(x) \neq 0$ for all positive integers n, then θ is continuous at x. If $f^n(x) = 0$, and n is minimal, then θ is discontinuous at x with discontinuities given by*

$$\theta(x+) = \theta(x) + t^n\nu(f),$$
$$\theta(x-) = \theta(x) - t^n\nu(f).$$

In particular, $\theta(x-) = -\nu(f)$.

It follows from Proposition 1.1 that for every $x \in I$, $\theta(x) = 0$ or else $\theta(x) \geqslant \nu(f)$ or $\theta(x) \leqslant -\nu(f)$. Applying this to $f^n(x)$ gives the following inequality that holds for a terminal segment $\sum_{i=n}^{\infty}\theta_i(x)t^i$ of $\theta(x)$:

$$(1)\qquad \begin{cases} \text{either} \quad \theta_n(x) \neq 0 \quad \text{and} \quad \left|\sum_{i=n}^{\infty}\theta_i(x)t^{i-n}\right| \geqslant \nu(f), \\[2ex] \text{or else} \quad \sum_{i=n}^{\infty}\theta_i(x)t^i = 0. \end{cases}$$

Here $|\sum_{i=n}^{\infty}\theta_i t^{i-n}|$ denotes $(\operatorname{sgn}\theta_n)(\theta_n + \theta_{n+1}t + \ldots)$. Any formal power series or polynomial with coefficients 0, $+1$, or -1 that satisfies (1) for every n is called a *$\nu(f)$-admissible* series. It follows immediately that for every x, $\theta(x+)$ and $\theta(x-)$ are also $\nu(f)$-admissible. The converse is also true:

PROPOSITION 1.3 (Milnor [4]). *For every $\nu(f)$-admissible series $\varphi \in \mathbf{Z}[[t]]$ there is a point $x \in I$ such that φ is equal to $\theta(x)$, $\theta(x-)$, or $\theta(x+)$.*

Of course $\nu = \nu(f) = \theta(0+)$ is also admissible with respect to itself. That is, for every integer $n \geqslant 0$,

$$|\nu_n + \nu_{n+1}t + \nu_{n+2}t^2 + \ldots| \geqslant \nu. \tag{2}$$

Any element of $\mathbf{Z}[[t]]$ with entries ± 1 satisfying (2) is called an *admissible kneading invariant.* Milnor shows in [4] that every admissible kneading invariant is the kneading invariant of a function of the type under consideration.

EXAMPLES. The formal power series $(1-t)^{-1}$ is easily seen to be an admissible kneading invariant, as is $(1-t)(1-t^2)(1-t^{2^2})\ldots(1-t^{2^{n-1}})(1-t^{2^n})^{-1}$ for every integer $n \geqslant 0$. These particular admissible kneading invariants will play an important role in the sequel. The formal power series $\gamma = 1+t-t^2+t^3-t^4+\ldots = 1+t(1-t)(1-t^2)^{-1}$ is not an admissible kneading invariant, for

$$|\gamma_1+\gamma_2 t+\gamma_3 t^2+\ldots| = 1-t+t^2-t^3+\ldots,$$

which is smaller than γ in the lexicographical ordering, as the t-coefficient in $|\gamma_1+\gamma_2 t+\gamma_3 t^2+\ldots|$ is less than the corresponding coefficient in γ. However, γ does qualify as a $(1-t)(1-t^2)^{-1}$-admissible series. On the other hand, γ is not $(1-t)^{-1}$-admissible.

2. Periodic orbits

In this section we look at the relationship between the periodicity of the orbit of a point x and the periodicity of the invariant coordinate $\theta(x)$.

An element $\varphi = \sum_{i=0}^{\infty} \varphi_i t^i$ of $\mathbf{Z}[[t]]$ is *periodic* if there is an integer $n > 0$ such that $\varphi_{i+n} = \varphi_i$ for all $i \geqslant 0$. φ is *anti-periodic* if there is an integer $n > 0$ such that $\varphi_{i+n} = -\varphi_i$ for all $i \geqslant 0$. By the *minimal period* of φ we mean the smallest $n > 0$, if there is one, for which φ is either periodic or anti-periodic. If φ is antiperiodic it is also periodic; however, the minimal period n of φ will then be an integer for which $\varphi_{i+n} = -\varphi_i$ for all $i \geqslant 0$. That is, φ is then antiperiodic (and not periodic) of minimal period n.

If an invariant coordinate $\theta(x)$, for $x \neq 0$, is periodic or anti-periodic, then $\theta(x)$ has no zero terms, and thus 0 is not in the orbit of x. From the relationship between $\theta(x)$ and $\varepsilon(x)$ it can be seen that if $\theta(x)$ has minimal period n, then $\varepsilon(x)$ is periodic of period n, except perhaps at the first term.

That is, $\varepsilon_i(x) = \varepsilon_{i+n}(x)$ provided $i \geqslant 1$, and n is the smallest positive integer with this property. Conversely, if $\varepsilon_i(x) = \varepsilon_{i+n}(x)$ for all $i \geqslant 1$, and if n is the smallest positive integer with this property, and if 0 is not in the orbit of x, then $\theta(x)$ is periodic or anti-periodic of minimal period n. In fact, then

$$\begin{aligned}\theta_{i+n}(x)/\theta_i(x) &= \varepsilon_{i+1}(x)\varepsilon_{i+2}(x)\ldots\varepsilon_{i+n}(x)\\ &= \varepsilon_1(x)\varepsilon_2(x)\ldots\varepsilon_n(x).\end{aligned}$$

We summarize this as follows:

LEMMA 2.0. *Where $x \neq 0$, $\theta(x)$ is periodic or anti-periodic of minima period n if and only if the orbit of x does not contain* 0 *and $\varepsilon_i(x) = \varepsilon_{i+n}(x)$ for all $i \geqslant 1$ and n is the smallest integer for which this is true. In this case, whether $\theta(x)$ is periodic or anti-periodic of period n is determined by the sign of $\varepsilon_1(x)\varepsilon_2(x)\ldots\varepsilon_n(x)$.*

It follows that if the orbit of x is periodic and does not contain 0, then $\varepsilon(x)$ and $\theta(x)$ are periodic. By the *minimal period* of a periodic point x, and of its orbit, we shall mean the smallest integer n such that $f^n(x) = x$.

We consider two periodic orbits *equivalent* if there are representative points x and y of the two orbits such that for all $k \geqslant 0$ the restriction of f^k to $C(x,y)$ is a homeomorphism. When this happens we will also call the points x and y equivalent, and write $x \sim y$. If $\mathcal{O}$ is a periodic orbit, we let $[\mathcal{O}]$ denote the equivalence class to which it belongs.

To prove the transitivity of this equivalence relation it suffices to show that if $x \sim y$, $z \sim w$, and $C(x,y) \cap C(z,w) \neq \emptyset$, then $x \sim y \sim z \sim w$. We prove this more general result because it will be needed again in Lemma 2.1 below. If $C(x,y) \cap C(z,w)$ has a non-empty interior this follows from the fact that for all $k \geqslant 0$ the restrictions $f^k | C(x,y)$ and $f^k | C(z,w)$ are monotone and that because of the overlap they must both be increasing or both decreasing. If $C(x,y) \cap C(z,w)$ consists of a single point, this must be an end point of each interval. Suppose, then, that $x < y = z < w$. Then for every $k \geqslant 0$ we have $f^k(C(x,y)) \cap f^k(C(y,w)) = \{f^k(y)\}$. For otherwise one of the intervals $f^k(C(x,y))$ and $f^k(C(y,w))$ is contained in the other, say $f^k(C(x,y)) \subset f^k(C(y,w))$, and then we may choose an integer $n \geqslant k$ such that $f^n(x) = x$, $f^n(y) = y$, and $f^n(w) = w$, and thus obtain

$$C(x,y) = f^n(C(x,y)) \subset f^n(C(y,w)) = C(y,w).$$

This contradicts the assumption that $C(x,y) \cap C(y,w) = \{y\}$, and so establishes the assertion that then $f^k(C(x,y)) \cap f^k(C(y,w)) = \{f^k(y)\}$ for every $k \geqslant 0$. It follows from this fact that $f^k | C(x,w)$ is a homeomorphism also in this case. We have now shown that the equivalence is transitive.

Suppose x is a periodic point. If the class of periodic points equivalent to x contains other points as well, let $[y, z]$ be the closed convex hull of this equivalence class.

LEMMA 2.1. *y and z are periodic and are equivalent to x. Moreover, if m is the minimal period among the periods of points equivalent to x, then $f^m|[y,z]$ is either a sense-preserving or a sense-reversing homeomorphism onto $[y,z]$. In the first case, all the periodic points equivalent to x have minimal period m. In the second case, there is precisely one point equivalent to x and of minimal period m, while the remaining points in the equivalence class have minimal period $2m$.*

Proof. There are sequences (y_i) and (z_i) such that $y_i \geqslant y$, $z_i \leqslant z$, $y_i \to y$, $z_i \to z$, and y_i and z_i are equivalent to x. Since $f^k|[y_i, z_i]$ is a homeomorphism for every i, so is $f^k|[y,z]$. Let m be the smallest positive integer such that $\operatorname{int}\{f^m([y,z]) \cap [y,z]\} \neq \emptyset$. Then, for sufficiently large i, $f^m([y_i, z_i]) \cap [y_i, z_i] \neq \emptyset$. That is, $f^m(y_i) \sim f^m(z_i)$, $y_i \sim z_i$, and

$$C(f^m(y_i), f^m(z_i)) \cap C(y_i, z_i) \neq \emptyset.$$

By the discussion preceding this lemma, this implies that

$$f^m(y_i) \sim f^m(z_i) \sim y_i \sim z_i.$$

Therefore $f^m(y_i), f^m(z_i) \in [y,z]$, and so $f^m[y,z] \subset [y,z]$. If this inclusion is strict, at least one of the y_i, z_i lies outside $f^m([y,z])$ and thus cannot be periodic. Therefore $f^m([y,z]) = [y,z]$. In particular, $y \sim x \sim z$.

Now if $f^m|[y,z]$ is sense-preserving, and $u \in [y,z]$, then either $f^m(u) = u$ or else $u, f^m(u), f^{2m}(u), \ldots$ is a monotone sequence. Thus all periodic points in $[y,z]$ have minimal period m. If $f^m|[y,z]$ is sense-reversing, then there is a unique point $w \in (y,z)$ fixed under f^m. Thus w has minimal period m. Any other point $u \in [y,z]$ either is non-periodic or has period at least $2m$. However, if $f^{2m}(u) \neq u$, then $u, f^{2m}(u), f^{4m}(u), \ldots$ is a monotone sequence. Thus every periodic point other than w has period $2m$ in this case.

If $\mathcal{O}$ is a periodic orbit not containing 0, we may associate to $\mathcal{O}$ the formal power series

$$\mu(\mathcal{O}) = \inf_{x \in \mathcal{O}} |\theta(x)|.$$

If $0 \in \mathcal{O}$ we let $\mu(\mathcal{O}) = \nu(f)$. Note that by (1) we have $|\theta(x)| \geqslant \nu(f)$ for all $x \in \mathcal{O}$, $x \neq 0$. Therefore $\mu(\mathcal{O}) \geqslant \nu(f)$.

We claim that $\mu(\mathcal{O})$ is an invariant of the equivalence class of periodic orbits. For suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct, but equivalent, periodic orbits. At least one of them, say $\mathcal{O}_2$, will not contain 0. Choose $y \in \mathcal{O}_1$, and $z \in \mathcal{O}_2$ equivalent to y, and choose $x \in \operatorname{int} C(y,z)$. Then $f^k(x) \neq 0$ for all $k \geqslant 0$, because $f^k | C(y,z)$ is a homeomorphism for all $k \geqslant 0$. Therefore, by Lemma 1.0, $\varepsilon(x) = \varepsilon(z)$. Hence also $\varepsilon(f^k(x)) = \varepsilon(f^k(z))$, and so $\theta(f^k(x)) = \theta(f^k(z))$ for all $k \geqslant 0$. Thus $\mu(\mathcal{O}_2) = \inf_{k \geqslant 0} |\theta(f^k(x))|$.

If also $\mathcal{O}_1$ does not contain 0 we get $\mu(\mathcal{O}_1) = \inf_{k \geqslant 0} |\theta(f^k(x))|$ as well, and thus $\mu(\mathcal{O}_1) = \mu(\mathcal{O}_2)$. If, however, $\mathcal{O}_1$ does contain 0, we may assume $y = 0$. Since we can let $x \in \operatorname{int} C(0,z)$ converge to 0, we must then have $\theta(x) = \pm \nu(f)$. That is, $|\theta(x)| = \nu(f)$ for every $x \in \operatorname{int} C(0,z)$. Since $\mu(\mathcal{O}_2) \geqslant \nu(f)$ for any orbit $\mathcal{O}_2$ it follows that $\mu(\mathcal{O}_2) = \nu(f) = \mu(\mathcal{O}_1)$.

LEMMA 2.2. *If $\mathcal{O}$ is a periodic orbit of period m, then $\mu(\mathcal{O})$ is periodic or anti-periodic of period m.*

Proof. First suppose $0 \notin \mathcal{O}$. Then $\varepsilon(x)$ is periodic with period m for every $x \in \mathcal{O}$. Then, by Lemma 2.0, $\theta(x)$ is periodic or anti-periodic with period m for every $x \in \mathcal{O}$, and therefore $\mu(\mathcal{O})$ is periodic or anti-periodic with period m.

If $0 \in \mathcal{O}$, then $f^m(0) = 0$. Let $N > 0$ be a large integer. Then $\bigcup_{n<N} f^{-n}(0)$ is a finite set, and therefore there must be intervals $J_1 = [-y, 0]$ and $J_2 = [0, y]$ on which f^k is a homeomorphism for all $k \leqslant N$. Suppose m is the minimal period of 0. By Lemma 1.0, $\varepsilon(x)$ is constant (mod t^N) on both $(-y, 0)$ and $(0, y)$. If i is not a multiple of m, then $f^i(0) \neq 0$, and therefore by letting $x_1 \in (0, y)$ and $x_2 \in (-y, 0)$ approach 0 we see that $\varepsilon_i(0) = \varepsilon_i(x_1) = \varepsilon_i(x_2)$ for all $x_1 \in (0,y)$ and $x_2 \in (-y, 0)$. On the other hand, if $f^m(J_2) \subset (-\infty, 0] \cap I$ then also $f^m(J_1) \subset (-\infty, 0] \cap I$, for $f(J_1)$ and $f(J_2)$ lie on the same side of $f(0)$. Similarly, if $f^m(J_1) \subset [0, \infty) \cap I$ then also $f^m(J_2) \subset [0, \infty) \cap I$. Thus for $x \in \operatorname{int} J_i$, with $i = 1, 2$,

$$\varepsilon_m(x) = \varepsilon_{2m}(x) = \ldots = \varepsilon_{km}(x)$$

provided $km < N$.

Therefore, if $x \in \operatorname{int} J_2$, we have $\varepsilon_{i+m}(x) = \varepsilon_i(x)$ if $i \geqslant 1$ and $i+m < N$. Therefore, by Lemma 2.0, $\theta_{i+m}(x) = \pm \theta_i(x)$ for $i \geqslant 0$ and $i+m < N$. That is, $\theta(x)$ is periodic or anti-periodic of period m and modulo t^N. Letting $x \in \operatorname{int} J_2$ converge to 0, we see that for $x \in \operatorname{int} J_2$, $\theta(x) = \nu(f)$ (mod t^N). Thus $\nu_i = \pm \nu_{i+m}$ provided $i + m < N$. Since N is arbitrary, this proves that $\mu(\mathcal{O}) = \nu(f)$ is periodic or anti-periodic with period m.

If $\mathcal{O}$ is a periodic orbit, then $\mu(\mathcal{O})$ is an admissible kneading invariant. Since $\mu(\mathcal{O}) \geqslant \nu(f)$, we only need to prove this for the case where $\mu(\mathcal{O}) > \nu(f)$. But then $0 \notin \mathcal{O}$. Therefore, if we let

$$\mu(\mathcal{O}) = |\theta(x)| = \left| \sum_{i=0}^{\infty} \theta_i(x) t^i \right|,$$

then by the definition of $\mu(\mathcal{O})$,

$$\mu(\mathcal{O}) \leqslant |\theta(f^n(x))| = \left|\theta_{n-1}(x)\left[\sum_{i=n}^{\infty} \theta_i(x)t^{i-n}\right]\right| = \left|\sum_{i=n}^{\infty} \theta_i(x)t^{i-n}\right|.$$

We also have the following converse:

PROPOSITION 2.3. *Let $\tilde{\theta}$ be a periodic admissible kneading invariant of period m such that $\tilde{\theta} \geqslant \nu(f)$. Then there is an equivalence class $[\mathcal{O}]$ of periodic orbits such that $\mu(\mathcal{O}) = \tilde{\theta}$.*

Proof. It follows immediately from the definitions that if $\tilde{\theta}$ is an admissible kneading invariant and $\tilde{\theta} \geqslant \nu(f)$, then $\tilde{\theta}$ is a $\nu(f)$-admissible formal power series.

By Proposition 1.3 there is a point $x \in I$ such that $\tilde{\theta}$ equals $\theta(x)$, $\theta(x-)$, or $\theta(x+)$. If we can find a point w with these properties and such that w is also periodic, we may complete the proof as follows. If θ is continuous at w, $\theta(w) = \tilde{\theta}$ and so we choose $\mathcal{O}$ to be the orbit of w. By Proposition 1.2, $0 \notin \mathcal{O}$. Since for any $l \geqslant 0$ we have $|\tilde{\theta}_l + \tilde{\theta}_{l+1}t + \tilde{\theta}_{l+2}t^2 + \dots| \geqslant \tilde{\theta}$, and so

$$|\theta(f^l(w))| = |\theta_{l-1}(w)[\theta_l(w) + \theta_{l+1}(w)t + \theta_{l+2}(w)t^2 + \dots]| \geqslant \tilde{\theta}.$$

Therefore $\mu(\mathcal{O}) = \tilde{\theta}$. On the other hand, if θ is not continuous at w, then by Proposition 1.2 we conclude that $f^k(w) = 0$. That is, 0 is in the orbit $\mathcal{O}$ of w. $\tilde{\theta} = \theta(w\pm)$ implies

$$\tilde{\theta} = \theta(w) \pm t^k\nu(f),$$

and thus $\nu(f) \geqslant \tilde{\theta}$, since $\tilde{\theta}$ is an admissible kneading invariant. Combining this with $\nu(f) \leqslant \tilde{\theta}$, we see that if 0 is in the orbit of w, then $\tilde{\theta} = \nu(f) = \mu(\mathcal{O})$. Thus it suffices to find a point w such that $\tilde{\theta}$ equals one of $\theta(w)$, $\theta(w-)$, $\theta(w+)$, and such that w is periodic.

If the point x, found above, is not already periodic, we find the point w as follows. Choose a large integer $N > m$ and a point y near x such that $\theta(y) = \tilde{\theta} \pmod{t^N}$. If $\theta(x) = \tilde{\theta}$ we may choose $y = x$. Then

$$\begin{aligned}\theta(f^m(y)) &= \theta_{m-1}(y)[\theta_m(y) + \theta_{m+1}(y)t + \dots] \\ &= \tilde{\theta}_{m-1}[\tilde{\theta}_m + \tilde{\theta}_{m+1}t + \dots] \pmod{t^{N-m}} \\ &= \tilde{\theta}_{m-1}.\tilde{\theta} \pmod{t^{N-m}}.\end{aligned}$$

Similarly, if $km < N$,

$$\begin{aligned}\theta(f^{km}(y)) &= \tilde{\theta}_{km-1}[\tilde{\theta}_{km} + \tilde{\theta}_{km+1}t + \dots] \pmod{t^{N-km}} \\ &= \tilde{\theta}_{km-1}.\tilde{\theta} \pmod{t^{N-km}} \\ &= \tilde{\theta}_{m-1}.\tilde{\theta} \pmod{t^{N-km}}.\end{aligned}$$

Now letting $y \to x$ and $N \to \infty$ we see that for each integer $k \geqslant 1$, one of $\theta(f^{km}(x))$, $\theta(f^{km}(x)-)$, and $\theta(f^{km}(x)+)$ is equal to $\alpha\tilde{\theta}$, where $\alpha = \tilde{\theta}_{m-1}$. Let K be the closed convex hull of the set $\{f^{km}(x)\}_{k=1,2,\ldots}$. It follows from the preceding observations together with the fact that θ is monotone, that for every $u \in \operatorname{int} K$ we have $\theta(u) = \alpha\tilde{\theta}$. Then $\varepsilon(u)$ is also constant on $\operatorname{int} K$. Therefore, by Lemma 1.0, $f^m|K$ is a homeomorphism on K, and since the set $\{f^{km}(x)\}_{k=1,2,\ldots}$ is invariant under f^m, f^m maps K onto itself. Now let w be the smaller boundary point of K. Then w is periodic, and $\tilde{\theta}$ equals one of $\theta(w-)$, $\theta(w+)$, and $\theta(w)$.

The main theorem of this section is the following:

THEOREM 2.4. *μ induces an injective map from the set of equivalence classes of periodic orbits onto the set of periodic and anti-periodic admissible kneading invariants greater than or equal to $\nu(f)$.*

Proof. Lemma 2.2, together with the remarks following it, shows that μ maps into the set of periodic and anti-periodic admissible kneading invariants greater than or equal to $\nu(f)$. Proposition 2.3 shows that μ is surjective. It only remains to prove that μ is injective.

Suppose $\mathcal{O}$ and $\mathcal{O}'$ are distinct periodic orbits such that $\mu(\mathcal{O}) = \mu(\mathcal{O}') = \tilde{\theta}$. Let m be the larger of their minimal periods. At least one of them, say $\mathcal{O}'$, does not contain 0. If $0 \notin \mathcal{O}$, choose $x \in \mathcal{O}$ in such a way that $|\theta(x)| = \mu(\mathcal{O}) = \tilde{\theta}$. If $0 \in \mathcal{O}$, let $x = 0$. Choose $y \in \mathcal{O}'$ so that

$$|\theta(y)| = \mu(\mathcal{O}') = \tilde{\theta}.$$

For a large integer $N > m$, let $u \in I$ be near x and such that $|\theta(u)| = \mu(\mathcal{O}) = \tilde{\theta} \pmod{t^N}$, and such that $u \notin \bigcup_{n<N} f^{-n}(0)$. If $0 \notin \mathcal{O}$ this can be achieved by simply putting $u = x$. Thus, for $i < N$,

$$\tilde{\theta}_i = \theta_0(u)\theta_i(u) = \theta_0(y)\theta_i(y).$$

Therefore, for $1 \leqslant i < N$,

$$\begin{aligned}\varepsilon_i(u) = \theta_i(u)/\theta_{i-1}(u) &= \theta_0(u)\theta_i(u)/\theta_0(u)\theta_{i-1}(u)\\ &= \theta_0(y)\theta_i(y)/\theta_0(y)\theta_{i-1}(y) = \varepsilon_i(y).\end{aligned}$$

Therefore, by Lemma 1.0, $f^k|C(f^m(u), f^m(y))$ is a homeomorphism if $0 \leqslant k \leqslant N-m$. Now let $u \to x$ and $N \to \infty$. Thus $f^k|C(x,y)$ is a homeomorphism for all $k \geqslant 0$. This proves that $\mathcal{O}$ and $\mathcal{O}'$ are equivalent.

If f is differentiable on $I \setminus \{0\}$ and satisfies $|df/dx| > 1$, as in Fig.2, then for any interval $[x,y]$ there is an integer $N > 0$ such that $0 \in f^N[x,y]$. Hence $\theta_N(x) \neq \theta_N(y)$. Thus there is no interval on which θ is constant. In particular, equivalent periodic orbits are identical for such functions and

μ is a one-to-one correspondence between periodic orbits and periodic kneading invariants greater than or equal to $\nu(f)$.

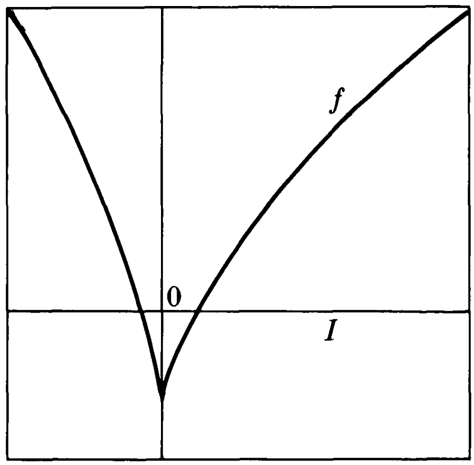

FIG. 2

By the *minimal period* of an equivalence class of orbits we mean the minimum among the minimal periods of orbits belonging to the equivalence class. In Lemma 2.1 we saw that if n is the minimal period of an equivalence class, then either all orbits in the equivalence class have minimal period n, or else there is one orbit with minimal period n, while all others have minimal period $2n$.

PROPOSITION 2.5. *The minimal period of an equivalence class* $[\mathcal{O}]$ *of periodic orbits is equal to the minimal period of* $\mu(\mathcal{O})$.

Proof. Let n be the minimal period of the equivalence class. By Lemma 2.2, $\mu(\mathcal{O})$ is periodic or anti-periodic of period n. Suppose the minimal period of $\mu(\mathcal{O})$ is l where $n = lr$, for $r > 1$. Let N be a large integer. Choose a point $x_0 \in \mathcal{O}$ and a point x near x_0 in such a way that $|\theta(x)| = \mu(\mathcal{O}) \pmod{t^N}$. Then $\theta_{i+l}(x) = \pm\,\theta_i(x)$ for $i+l < N$. Therefore, by Lemma 2.0, $\varepsilon_{i+l}(x) = \varepsilon_i(x)$ for $i+l < N$ and $i \geqslant 1$. Hence

$$\varepsilon_i(f^{n+l}(x)) = \varepsilon_i(f^n(x))$$

for $0 \leqslant i < N-n-l$. Let $y = f^n(x)$. Then by Lemma 1.0, either $f^l(y) = y$, or $f^k | C(y, f^l(y))$ is a homeomorphism for all $k \leqslant N-m-l$. Letting $x \to x_0$ and $N \to \infty$, and thus also $y \to x_0$, we conclude that either $f^l(x_0) = x_0$ or $f^k | C(x_0, f^l(x_0))$ is a homeomorphism for all k. But $f^l(x_0) = x_0$ is impossible since the minimal period of x_0 is no less than $n > l$. If $f^l | C(x_0, f^l(x_0))$ preserves sense, the sequence x_0, $f^l(x_0)$, $f^{2l}(x_0)$, $f^{3l}(x_0)$, ...

is monotone. Since x_0 is periodic this is impossible. Thus $f^l \mid C(x_0, f^l(x_0))$ reverses the sense. Therefore either $f^{2l}(x_0) = x_0$ or else the sequence $x_0, f^{2l}(x_0), f^{4l}(x_0), \ldots$ is monotone. If $f^{2l}(x_0) = x_0$ then $\operatorname{int} C(x_0, f^l(x_0))$ contains a point equivalent to x_0 of minimal period l. This contradicts the assumption that $n > l$ is the minimal period of the equivalence class of periodic orbits. On the other hand, the sequence $x_0, f^{2l}(x_0), f^{4l}(x_0), \ldots$ cannot be monotone, because x_0 is periodic. Thus the assumption that the minimal period of $\mu(\mathcal{O})$ is less than n leads to a contradiction.

3. Periodic admissible kneading invariants

Now that we have demonstrated a close relationship between the existence of periodic orbits and the existence of periodic and anti-periodic admissible kneading invariants larger than $\nu(f)$, we will study the existence of the latter. In particular, we shall be interested to find the largest periodic or anti-periodic admissible kneading invariant of each period.

To facilitate our discussion we shall use a simpler notation for formal power series with coefficients ± 1. If $\sum_{i=0}^{\infty} \theta_i t^i$ is such a power series we shall also denote it by $\theta_0 \theta_1 \theta_2 \ldots$. Thus $++-+-+-\ldots$ represents the formal power series $1+t-t^2+t^3-t^4+t^5-t^6+\ldots$. Furthermore, if α represents a finite series (that is, a polynomial) then α' will denote the infinite series produced by iterating α indefinitely. Finally, if α denotes a series (finite or infinite), $\bar{\alpha}$ will denote the power series obtained by changing all the signs.

Definition. We let $\alpha(\frac{1}{2})$ denote the symbol $+$. Thus

$$\alpha(\tfrac{1}{2})' = +++\ldots = 1+t+t^2+\ldots.$$

We now define inductively, for $n \geqslant 0$,

$$\alpha(2^n) = \alpha(2^{n-1})\overline{\alpha(2^{n-1})}.$$

Thus $\alpha(1) = +-$ and $\alpha(2) = +--+$, etc. It is easy to check that for $n \geqslant 0$, $\alpha(2^n)'$ is an admissible kneading invariant which is anti-periodic of minimal period 2^n.

Lemma 3.0. *Suppose ν is an admissible kneading invariant and suppose α is a finite sequence of $+$'s and $-$'s beginning with a $+$. If ν begins $\alpha\bar{\alpha}+\ldots$, then $\nu = (\alpha\bar{\alpha})'$.*

Proof. Let k be the length of the sequence α. Suppose $\nu = \alpha\bar{\alpha}\sigma\ldots$, where σ has length k. Using the fact that ν is an admissible kneading invariant we compare ν with a part of itself as indicated in the following diagram:

$$\left.\begin{array}{l} \nu = \alpha\bar{\alpha}\sigma\ldots \\ \nu = \quad\alpha\bar{\alpha}\ldots \end{array}\right\}.$$

That is, we use (2) to compare ν with $|\nu_k + \nu_{k+1}t + \nu_{k+2}t^2 + \dots|$, and conclude that $\bar{\sigma} \geqslant \bar{\alpha}$. That is, $\sigma \leqslant \alpha$. Now compare again as follows:

$$\left.\begin{array}{l} \nu = \alpha\bar{\alpha}\sigma\dots \\ \nu = \quad \alpha\dots \end{array}\right\}.$$

Since σ begins with a $+$, we see that $\sigma \geqslant \alpha$. Thus $\sigma = \alpha$. It is now easy to show by induction, using comparisons of this kind, that ν consists entirely of a pattern alternating α and $\bar{\alpha}$.

PROPOSITION 3.1. *If ν is an admissible kneading invariant such that $\nu > \alpha(2^n)'$, then ν equals $\alpha(2^m)'$ for some $m < n$. Thus for $n \geqslant 1$, $\alpha(2^n)'$ is the maximum element in the set of periodic or anti-periodic kneading invariants of minimal period 2^n.*

Proof. For $n = 0, -1$ the theorem is clearly true. We now assume the theorem proved for $n = k$, and prove it for $n = k+1$. If $\nu > \alpha(2^k)'$ the result follows from the induction hypothesis, so we may assume that the first block of 2^{k+1} entries of ν equals $\alpha(2^k)$ and the next block σ of length 2^{k+1} is larger than $\overline{\alpha(2^k)}$. We may assume that σ begins with $-$, for otherwise we can complete the proof by invoking Lemma 3.0. Now compare

$$\left.\begin{array}{l} \nu = \alpha(2^n)\sigma\dots \\ \nu = \qquad \alpha(2^n)\dots \end{array}\right\}$$

to obtain $\bar{\sigma} \geqslant \alpha(2^n)$. That is, $\sigma \leqslant \overline{\alpha(2^n)}$ contrary to our hypothesis.

Having shown that $\alpha(2^n)'$ is the maximal admissible kneading invariant of minimal period 2^n we want to identify the maximal periodic or anti-periodic admissible kneading invariants for all other periods.

DEFINITION. If k is odd and greater than 1, and $n \geqslant 0$, let

$$\alpha(2^n.k) = \alpha(2^{n-1})\overline{\alpha(2^{n-1})}\,\overline{\alpha(2^{n-1})}\alpha(2^n)\dots\alpha(2^n),$$

where $\alpha(2^n)$ is repeated $\frac{1}{2}(k-3)$ times making $2^n.k$ the total length of $\alpha(2^n.k)$. We will show in Theorem 3.4 below that $\alpha(2^n.k)'$ is the maximum among periodic or anti-periodic admissible kneading invariants of minimal period $2^n.k$.

LEMMA 3.2. *Let ν be an admissible kneading invariant and let $\beta = \alpha(2^{k-1})$. Suppose ν begins $\beta\bar{\beta}\bar{\beta}\beta\dots$. Then ν is entirely a succession of blocks of length 2^k each equal to β or $\bar{\beta}$.*

Proof. The proof is by induction. The theorem is certainly true if $k = 0$. Suppose we have shown the theorem to be true for an admissible kneading invariant whose first 2^{k+1} terms are $\gamma\bar{\gamma}\bar{\gamma}\gamma$ where $\gamma = \alpha(2^{k-2})$. Note that $\beta\bar{\beta}\bar{\beta}\beta = \gamma\bar{\gamma}\bar{\gamma}\gamma\bar{\gamma}\gamma\gamma\bar{\gamma}$. Partition ν into a succession of blocks of length 2^k. Suppose σ is the first such block that is not equal to β or $\bar{\beta}$. By the induction hypothesis, σ equals one of the following: $\gamma\gamma$ or $\bar{\gamma}\bar{\gamma}$. To complete the proof, consider four cases: σ is preceded by $\beta\beta$, by $\bar{\beta}\bar{\beta}$, by $\beta\bar{\beta}$, or by $\bar{\beta}\beta$. In each case, two comparisons will result in a contradiction to the assumption that ν is an admissible kneading invariant.

LEMMA 3.3. *Let μ be an admissible kneading invariant. Suppose $\mu \geqslant \alpha(2^n.k)'$, with k odd and greater than* 1. *Then either $\mu = \alpha(2^m)'$ for some $m \leqslant n$, or else the first $2^n(k+1)$ terms of μ consist of a succession of blocks each equal to $\alpha(2^n)$ or $\overline{\alpha(2^n)}$, while the first 4.2^n terms are precisely* $\alpha(2^{n-1})\overline{\alpha(2^{n-1})}\,\overline{\alpha(2^{n-1})}\alpha(2^{n-1})$.

Proof. We will assume that μ is not equal to $\alpha(2^m)'$ for any $m \leqslant n$ and then prove the alternative. Let $\beta = \alpha(2^{n-1})$ and $\alpha = \alpha(2^n)$ and $\nu = \alpha(2^n.k)'$. We first show that μ begins $\beta\bar{\beta}\bar{\beta}\beta\ldots$.

By Proposition 3.1, μ certainly begins with $\beta\bar{\beta}\ldots$. Let σ be the next block of length 2^n. Compare

$$\left.\begin{aligned}\mu &= \beta\bar{\beta}\sigma\ldots\\ \mu &= \quad\ \beta\bar{\beta}\ldots\end{aligned}\right\}.$$

By Lemma 3.0 we know that σ begins with $-$. Hence this comparison shows $\bar{\sigma} \geqslant \beta$. That is, $\sigma \leqslant \bar{\beta}$. But combined with $\mu \geqslant \nu$ this implies $\sigma = \bar{\beta}$ and so μ begins with $\beta\bar{\beta}\bar{\beta}\ldots$. But then, because $\mu \geqslant \nu$, the block consisting of the first 4.2^n terms of μ must be $\geqslant \beta\bar{\beta}\bar{\beta}\beta$. Now Proposition 3.1 implies that $\mu = \beta\bar{\beta}\bar{\beta}\beta\ldots$.

If $k = 3$, this completes the proof of the lemma. In any case, by Lemma 3.2, every subsequent block of length 2^n equals β or $\bar{\beta}$. Note that the first $\frac{1}{2}(k+1)$ blocks of length 2^{n+1} in ν are

$$\alpha\bar{\alpha}\bar{\alpha}\ldots\bar{\alpha}.$$

Let σ be the first block in μ of length 2^{n+1} which is not equal to α or $\bar{\alpha}$, and suppose σ is contained within the first $2^n(k+1)$ terms. If σ is immediately preceded by α we look at the complete string of α's preceding σ (Case 1). Otherwise we look at the complete string of $\bar{\alpha}$'s preceding σ (Case 2). Since $\mu = \beta\bar{\beta}\bar{\beta}\beta\ldots = \alpha\bar{\alpha}\ldots$ we know that neither string includes the initial block of length 2^{n+1} in μ.

Case 1. If $\sigma = \beta\beta$, then $\alpha\sigma = \beta\bar{\beta}\beta\beta$, and so μ contains a succession $\bar{\beta}\beta\beta$. This is impossible, for the following comparison,

$$\left.\begin{aligned}\mu &= \ldots\beta\bar{\beta}\beta\beta\ldots\\ \mu &= \quad\ \beta\bar{\beta}\bar{\beta}\beta\ldots\end{aligned}\right\}$$

would then imply that $\bar{\beta} \geqslant \beta$, which is certainly false. Thus, to avoid a succession $\bar{\beta}\bar{\beta}\bar{\beta}$ we must have $\sigma = \beta\beta$ in this case. Now note that $\nu \leqslant \mu \leqslant |\mu_k + \mu_{k+1}t + \mu_{k+2}t^2 + \ldots|$. Therefore, we may compare

$$\left.\begin{array}{rl} \mu = & \ldots\bar{\alpha}\alpha\alpha\alpha\ldots\alpha\sigma\ldots \\ \nu = & \alpha\bar{\alpha}\bar{\alpha}\bar{\alpha}\ldots\bar{\alpha}\bar{\alpha}\ldots \end{array}\right\},$$

and conclude that $\bar{\sigma} \geqslant \bar{\alpha}$ and so $\sigma \leqslant \alpha$. But that contradicts $\sigma = \beta\beta$.

Case 2 is dismissed in the same way.

THEOREM 3.4. *For any integer $r > 1$, $\alpha(r)'$ is the maximum element in the set of all periodic and antiperiodic admissible kneading invariants of minimal period r.*

Proof. If r is a power of 2 this has already been proved in Proposition 3.1. Otherwise, $r = 2^n.k$, with k odd and greater than 1. Suppose μ is an admissible kneading invariant of minimal period r such that $\mu \geqslant \alpha(2^n.k)'$. Thus μ satisfies Lemma 3.3.

First suppose μ is periodic of period r. Then the second half of the $\frac{1}{2}(k+1)$th block σ of length 2^{n+1} begins with the rth term of the series μ, and thus equals $\beta = \alpha(2^{n-1})$. Therefore, by Lemma 3.3, σ equals $\bar{\alpha}$ where $\alpha = \alpha(2^n)$. This $\bar{\alpha}$ is preceded by a string of $\bar{\alpha}$'s (possibly empty), which is preceded by an α. If this α goes back to the first term, μ and $\nu = \alpha(2^n.k)'$ coincide on the first $(k+1)2^n$ terms, and so $\mu = \nu$. To show that α must go back to the first term, we suppose it does not. Then we note that, since μ is periodic, σ is followed by $\bar{\beta}\bar{\beta}$, and we compare the first terms of ν with the part of μ consisting of the succession $\alpha\bar{\alpha}\bar{\alpha}\ldots\bar{\alpha}\bar{\beta}\bar{\beta}$. That is,

$$\left.\begin{array}{rl} \mu = & \ldots\beta\bar{\beta}\bar{\beta}\beta\bar{\beta}\beta\ldots\bar{\beta}\beta\bar{\beta}\bar{\beta}\ldots \\ \nu = & \beta\bar{\beta}\bar{\beta}\beta\bar{\beta}\beta\ldots\bar{\beta}\beta\bar{\beta}\beta\ldots \end{array}\right\}.$$

This comparison shows that $\bar{\beta} \geqslant \beta$, which is certainly false.

The proof is similar in the case where μ is anti-periodic of period r, excluding this possibility altogether.

THEOREM 3.5. *The maximal periodic or anti-periodic admissible kneading invariants found in Theorem 3.4 exhibit the following order:*

$$\alpha(3) < \alpha(5) < \alpha(7) < \ldots < \alpha(2.3) < \alpha(2.5) < \ldots < \alpha(2^2.3) < \ldots < \alpha(2^n)$$
$$< \alpha(2^{n-1}) < \ldots < \alpha(4) < \alpha(2) < \alpha(1).$$

That is, $\alpha(m) < \alpha(n)$ if and only if $m \lhd n$.

The obvious proof is left to the reader.

THEOREM A. *Suppose $k \lhd l$. If f has a periodic orbit of minimal period k then it has one of minimal period l.*

Proof. Note that if we can show $\alpha(l) \geqslant \nu(f)$, Propositions 2.3 and 2.5 will imply the existence of an orbit of minimal period l. But by Lemma 2.1 and Proposition 2.5 if f has an orbit $\mathcal{O}$ of period k then $\mu(\mathcal{O})$ has minimal period k or $\frac{1}{2}k$. First suppose k is not a power of 2. Then $\alpha(k) > \alpha(\frac{1}{2}k)$ and so in either case

$$\nu(f) \leqslant \mu(\mathcal{O}) \leqslant \alpha(k) < \alpha(l).$$

If, on the other hand, $k = 2^n$ then $k \lhd l$ implies $l = 2^m$, $m < n$, and $\alpha(\frac{1}{2}k) \leqslant \alpha(l)$. Therefore also in this case $\nu(f) \leqslant \mu(\mathcal{O}) \leqslant \alpha(\frac{1}{2}k) \leqslant \alpha(l)$.

The correspondence between periodic orbits and periodic and anti-periodic admissible kneading invariants can be exploited to get estimates on the number of periodic orbits of a given period. Theorem B is an example of such a theorem. To prove it we need the following lemma:

LEMMA 3.6. *Let $\alpha = +-$ and suppose $\sigma_i = \alpha$ or $\bar{\alpha}$ for each $i = 1, \ldots, k$. Let*

$$\gamma = +--\underbrace{\alpha\alpha\ldots\alpha}_{r}\sigma_1\sigma_2\ldots\sigma_k.$$

Then γ' and $(\gamma\bar{\gamma})'$ are admissible kneading invariants larger than $\alpha(2r+1)'$.

Proof. To prove that γ' is an admissible kneading invariant we have to show that every comparison of γ' with itself succeeds. That is, we must show that if $\gamma' = \sum_{i=0}^{\infty} \gamma_i t^i$, then for every n we have the inequality (2):

$$\left| \sum_{i=0}^{\infty} \gamma_{n+i} t^i \right| \geqslant \gamma'.$$

Since the first three terms of γ' are $+--$, if $\gamma_n \gamma_{n+1} \gamma_{n+2}$ is one of the sequences $+++$, $---$, $++-$, $--+$, $+-+$, or $-+-$, the inequality (2) is established by merely comparing these three terms with the first three terms of γ'. Thus the only values of n for which the inequality (2) could conceivably fail are those for which $\gamma_n \gamma_{n+1} \gamma_{n+2} = +--$ or $-++$. The same goes for the series $(\gamma\bar{\gamma})'$. From now on we will only consider comparisons for such values of n.

Note that the first $4r+2k+5$ terms of γ' may be written as

$$\gamma' = +--\alpha\ldots\alpha\sigma_1\ldots\sigma_k\sigma_{k+1}\ldots\sigma_{k+r+1}\ldots,$$

where $\sigma_{k+1} = \alpha$ and $\sigma_{k+j} = \bar{\alpha}$ for $2 \leqslant j \leqslant r+1$. A triple $+--$ or $-++$ occurs in this list only where there is a sequence $\alpha\bar{\alpha} = +--+$ or a sequence $\bar{\alpha}\alpha = -++-$. So suppose that for some i, with $1 \leqslant i \leqslant k+1$,

$\sigma_i = \alpha$ is preceded by $\bar{\alpha}$. We compare γ' with $|\sum_{j=0}^{\infty} \gamma_{2(r+i)-1+j} t^j|$:

$$\left.\begin{aligned} \gamma' &= \ldots\bar{\alpha}\alpha\ldots\sigma_h\sigma_{h+1}\cdots \\ \gamma' &= \quad \alpha\underbrace{\bar{\alpha}\ldots\bar{\alpha}}_{r}-\,-\,\ldots \end{aligned}\right\},$$

where $h = i+r-1$, and so $r \leqslant h \leqslant k+r$. Certainly $\bar{\alpha}\ldots\bar{\sigma}_h \geqslant \bar{\alpha}\ldots\bar{\alpha}$, and $\bar{\sigma}_{h+1} > -\,-$. Therefore, this comparison succeeds. Precisely the same argument applies if for some i, with $1 \leqslant i \leqslant k$, $\sigma_i = \bar{\alpha}$ and is preceded by α. This proves that $|\sum_{i=0}^{\infty} \gamma_{i+n} t^i| \geqslant \gamma'$ if $n < 2r+2k+3$. Since γ' is periodic of period $2r+2k+3$ it follows that this inequality holds for every value of n. Thus γ' is an admissible kneading invariant.

Analogously, the first $4r+2k+5$ terms of $(\gamma\bar{\gamma})'$ may be written as

$$(\gamma\bar{\gamma})' = +\,-\,-\underbrace{\alpha\ldots\alpha}_{r}\sigma_1\ldots\sigma_k\sigma_{k+1}\cdots\sigma_{k+r+1}\cdots,$$

where $\sigma_{k+1} = \bar{\alpha}$ and $\sigma_{k+j} = \alpha$ for $2 \leqslant j \leqslant r+1$. The argument used above for γ' will now serve to prove that $(\gamma\bar{\gamma})'$ is also an admissible kneading invariant.

To prove that γ' and $(\gamma\bar{\gamma})'$ are greater than $\alpha(2r+1)'$, note that for both γ' and $(\gamma\bar{\gamma})'$ the first $4r+5$ terms may be written as

$$+\,-\,-\underbrace{\alpha\ldots\alpha}_{r}\sigma_1\ldots\sigma_r\sigma_{r+1}\sigma_{r+2},$$

where $\sigma_j = \alpha$ or $\bar{\alpha}$ for $1 \leqslant j \leqslant r+2$, and where at least one of the σ_j equals α. On the other hand, the first $4r+5$ terms of $\alpha(2r+1)'$ are

$$+\,-\,-\underbrace{\alpha\ldots\alpha}_{r}\underbrace{\bar{\alpha}\ldots\bar{\alpha}}_{r}-\,-,$$

From this it is clear that γ' and $(\gamma\bar{\gamma})'$ are greater than $\alpha(2r+1)'$.

THEOREM B. *Let n and l be odd numbers such that $n > l \geqslant 3$. Suppose f has an orbit of minimal period l. Then f has at least $2^{(n-l)/2}$ distinct orbits of minimal period n.*

Proof. Lemma 3.6 provides us with 2^{k+1} periodic or anti-periodic admissible kneading invariants of minimal period $2r+2k+3$, each of them strictly greater than $\alpha(2r+1)'$. Now let $l = 2r+1$ and $n = 2r+2k+3$. Thus the number of admissible kneading invariants provided by Lemma 3.6 is $2^{(n-l)/2}$. Combining this fact with Theorem 2.4 and Proposition 2.5, we see that there are at least $2^{(n-l)/2}$ equivalence classes of periodic orbits of minimal period n.

In fact, if l, n are odd and $n > l \geqslant 3$, the proof of Lemma 3.6 may be generalized to give us $2^{(n-l)/2}$ periodic or anti-periodic admissible kneading invariants of minimal period $n.2^p$, all of them strictly greater than $\alpha(l.2^p)'$. Thus the following more general version of Theorem B is true:

Let n and l be odd numbers such that $n > l \geqslant 3$. Suppose f has an orbit of period $l.2^p$. Then f has at least $2^{(n-l)/2}$ distinct orbits of period $n.2^p$.

4. Asymptotically periodic orbits

In this section we consider the relationship between asymptotically periodic behaviour of x and eventual periodicity of $\theta(x)$. x is *asymptotically periodic* if its orbit converges towards a periodic orbit; that is, if there is a periodic point y such that $\lim_{n\to\infty}|f^n(x)-f^n(y)| = 0$. We will also express this by saying that the orbit of x is asymptotic to the orbit of y. $\theta = \sum_{i=0}^{\infty}\theta_i t^i$ is *eventually periodic* if for some n the series $\sum_{i=0}^{\infty}\theta_{n+i}t^i$ is periodic and non-vanishing. If $\theta(x)$ is eventually periodic, and if $\mathcal{O}$ is the orbit of x, we can extend the definition of μ to such orbits by letting

$$\mu(\mathcal{O}) = \lim_{n\to\infty}\inf_{m\geqslant 0}|\theta(f^{n+m}(x))|.$$

As n gets larger, the set over which the infimum is taken decreases. Thus the limit as $n \to \infty$ is taken over a monotone increasing sequence of formal power series with coefficients -1, 0, or $+1$. This ensures that this limit exists.

The key lemma is the following.

LEMMA 4.0 (Milnor [4]). *If $\theta(x)$ is eventually periodic, then the orbit $\mathcal{O}$ of x is asymptotic to a periodic orbit $\mathcal{O}'$ such that $\mu(\mathcal{O}) = \mu(\mathcal{O}')$. Conversely, if the orbit $\mathcal{O}$ of x does not contain 0, and is asymptotic to a periodic orbit $\mathcal{O}'$, then $\theta(x)$ is eventually periodic and $\mu(\mathcal{O}) = \mu(\mathcal{O}')$.*

Proof. Suppose $\mathcal{O}$ is asymptotic to $\mathcal{O}'$ where $\mathcal{O}'$ and $\mu(\mathcal{O}')$ are periodic of period n. If $0 \notin \mathcal{O}'$, choose $y \in \mathcal{O}'$ so that $|\theta(y)| = \mu(\mathcal{O}')$. If $0 \in \mathcal{O}'$ choose $y = 0$. Then for some $z \in \mathcal{O}$, $f^{qn}(z) \to y$ as $q \to \infty$. If $0 \notin \mathcal{O}'$ it follows from Proposition 1.2 that $\lim_{q\to\infty}\theta(f^{qn}(z)) = \pm\mu(\mathcal{O}')$. If $y = 0$, there is an interval (a, b) containing 0, such that $(a,b)\cap\{\bigcup_{m=1}^{n-1}f^{-m}(0)\} = \varnothing$. Therefore f^n is a homeomorphism on $(a, 0)$ and $(0, b)$, mapping both to the same side of 0. In fact we may choose a and b in such a way that $f^n(a, 0) = f^n(0, b)$. There is a large integer N such that $q \geqslant N$ implies $f^{qn}(z) \in (a, b)$. But then for all $q > N$, $f^{qn}(z)$ is in the same common image of $(a, 0)$ and $(0, b)$ under f^n. Thus either $f^{qn}(z) \in (a, 0)$ for all $q > N$, or else $f^{qn}(z) \in (0, b)$ for all $q > N$. Thus $\lim_{q\to\infty}\theta(f^{qn}(z))$ exists also in this case, and equals $\pm\mu(\mathcal{O}')$.

Now choose M so large that for all $q \geqslant M$, $\theta(f^{qn}(z)) = \varepsilon\mu(\mathcal{O}')$ (mod t^{2n}). Here $\varepsilon = \pm 1$. We will show that then $\theta(f^{qn}(z)) = \varepsilon\mu(\mathcal{O}')$. For suppose we have already shown $\theta(f^{qn}(z)) = \varepsilon\mu(\mathcal{O}') \pmod{t^{pn}}$ for all $q \geqslant M$. Let $\mu(\mathcal{O}') = \sum_{i=0}^{\infty} \mu_i t^i$. Then for $0 \leqslant i < (p-1)n$ and $q \geqslant M$ we have

$$\varepsilon\mu_{i+n} = \theta_{i+n}(f^{qn}(z)),$$

while for $0 \leqslant i < pn$ we have, because $\theta_{n-1}(f^{qn}(z)) \neq 0$,

$$\theta_{i+n}(f^{qn}(z)) = \theta_{n-1}(f^{qn}(z))\theta_i(f^{(q+1)n}(z)) = \theta_{n-1}(f^{qn}(z))\varepsilon\mu_i.$$

Since $\mu_i = \mu_{i+n}$ it follows that $\theta_{n-1}(f^{qn}(z)) = 1$. Therefore, the above equalities imply $\theta_{i+n}(f^{qn}(z)) = \varepsilon\mu_i = \varepsilon\mu_{i+n}$ for $0 \leqslant i < pn$. Thus $\theta(f^{qn}(z)) = \varepsilon\mu(\mathcal{O}') \pmod{t^{(p+1)n}}$ for all $q \geqslant M$. It now follows by induction that for $q \geqslant M$, $\theta(f^{qn}(z)) = \varepsilon\mu(\mathcal{O}')$. It follows immediately that $\theta(x)$ is eventually periodic and that $\mu(\mathcal{O}) = \mu(\mathcal{O}')$.

To prove the first half of the lemma, we suppose that $\theta(x)$ is eventually periodic with (eventual) period n. Then $\{\theta(f^{p+m}(x))\}_{m\geqslant 0}$ is a finite set for every p, and for sufficiently large p this set does not depend on p. Therefore, we can choose $z = f^m(x) \in \mathcal{O}$ such that $|\theta(z)| = \mu(\mathcal{O})$. Then $\theta(z)$ is periodic of period n and thus, by Lemma 2.0, so is $\varepsilon(z)$, after the first term. That is, $\varepsilon(f^{nq}(z))$ is independent of $q \geqslant 1$. Thus f^n is a homeomorphism of the convex hull K of $\{f^{nq}(z)\}_{q=1}^{\infty}$ into itself. Hence f^{2n} is certainly sense-preserving, and therefore $\{f^{2nq}(z)\}_{q=1}^{\infty}$ is a monotone sequence, converging to a limit point $w \in K$ of period $2n$. Let $\mathcal{O}'$ be the orbit of w. It follows from the first half of this proof that $\mu(\mathcal{O}) = \mu(\mathcal{O}')$.

LEMMA 4.1. *Suppose θ is a $\alpha(2^N)'$-admissible invariant coordinate. Suppose that for some $n \leqslant N$ there is a block τ of terms in θ which equals $\alpha(2^n)$ and which is followed by a $-$. Then τ is followed by $\overline{\alpha(2^n)}$. Similarly, for $\overline{\alpha(2^n)}+$ we get $\overline{\alpha(2^n)}\alpha(2^n)$.*

Proof. If $n = N$, a series containing $\alpha(2^n)-$ cannot be $\alpha(2^N)'$-admissible. Thus $n < N$. Then $\alpha(2^N)$ begins $\alpha(2^n)-+\dots$. Therefore

$$\theta = \dots\alpha(2^n)-+\dots = \dots\alpha(2^n)\overline{\alpha(1)}\dots.$$

Now make the induction assumption that $\theta = \dots\alpha(2^n)\overline{\alpha(2^k)}\dots$ where $k < n$. Then, noting that

$$\alpha(2^N) = \alpha(2^n)\overline{\alpha(2^k)}\alpha(2^k)\dots = \alpha(2^k)\overline{\alpha(2^k)}\dots,$$

we compare

$$\left.\begin{aligned} \theta &= \dots\alpha(2^n)\overline{\alpha(2^k)}\sigma\dots \\ \alpha(2^N) &= \quad\ \alpha(2^n)\overline{\alpha(2^k)}\alpha(2^k)\dots \end{aligned}\right\},$$

where σ is a block of length 2^{k+1}. Thus $\sigma \geqslant \alpha(2^k)$. We also compare

$$\left.\begin{aligned} \theta &= \dots\alpha(2^n)\overline{\alpha(2^k)}\sigma\dots \\ \alpha(2^N) &= \qquad\quad \alpha(2^k)\overline{\alpha(2^k)}\dots \end{aligned}\right\}$$

giving $\sigma \leqslant \alpha(2^k)$. Thus $\sigma = \alpha(2^k)$. Thus $\theta = \ldots\alpha(2^n)\overline{\alpha(2^{k+1})}\ldots$. The result now follows by induction.

LEMMA 4.2. *Suppose θ is a $\alpha(2^N)'$-admissible invariant coordinate. Suppose n is the largest integer ($n \leqslant N$) such that θ contains a block of coefficients equal to the block $\pm\alpha(2^n)$. Then θ is eventually equal to $\alpha(2^n)'$.*

Proof. By replacing θ by $-\theta$ if necessary, we may assume that θ contains a block of coefficients equal to $\alpha(2^n)$. The hypothesis assures us that nowhere in θ is there a block of coefficients equal to $\pm\alpha(2^m)$ with $m > n$. Hence, by Lemma 4.1, every block $\alpha(2^n)$ is followed by $+$ and every block $\overline{\alpha(2^n)}$ is followed by $-$. We claim that the part γ of θ that begins with a block equal to $\alpha(2^n)$ is just equal to $\alpha(2^n)'$. Suppose this is false. Divide γ into successive blocks of length 2^n. The first two are equal to $\alpha(2^{n-1})$ and $\overline{\alpha(2^{n-1})}$ respectively. Let σ be the first such block at which this alternating pattern $\alpha(2^{n-1})\overline{\alpha(2^{n-1})}\alpha(2^{n-1})\ldots$ fails. Say σ is preceded by $\overline{\alpha(2^{n-1})}$, the rest of the proof being the same in either case. Then σ is preceded by $\alpha(2^n)$. Hence σ starts with a $+$. Hence by Lemma 4.1, $\sigma = \alpha(2^{n-1})$. This contradicts our assumption about σ.

THEOREM C. *If $\nu(f) \geqslant \alpha(2^N)'$ for some N, then $\nu(f) = \alpha(2^n)'$ for some $n \leqslant N$. In this case there is one equivalence class of periodic orbits for each minimal period 2^m, where $m = 1,\ldots,n$, plus two distinct fixed points, corresponding to $\alpha(1)$ and $\alpha(\frac{1}{2})$, and there are no other periodic orbits. Every orbit is asymptotic to one of these periodic orbits.*

Proof. It follows from Proposition 3.1 that $\nu(f) = \alpha(2^n)'$ for some $n \leqslant N$. Then, using Theorem 2.4 and Proposition 2.5, we obtain the above list of equivalence classes of periodic orbits for f. But then Lemma 4.2 and Lemma 4.0 combine to show that every orbit is asymptotic to an orbit in this list.

Having shown that the structure of f is relatively simple when $\nu(f) \geqslant \alpha(2^N)'$ for some N, we turn to the case where $\nu(f) < \alpha(2^N)'$ for all integers $N \geqslant 0$. Let $\lambda = \inf_N \alpha(2^N)'$. That is, define a formal power series λ by requiring that for every $N > 0$, $\lambda = \alpha(2^N)' \pmod{t^{2^{N+1}}}$. It is easy to check that λ is an admissible kneading invariant. λ defines a dividing line at which the complexity of the structure of f increases dramatically.

PROPOSITION 4.3. *If μ is an admissible kneading invariant and $\mu < \alpha(2^N)'$ for all $N \geqslant 0$, then either $\mu = \lambda$ or else $\mu \leqslant \alpha(2^n.k)'$ for some n and some odd $k \geqslant 3$.*

Proof. If for some n, $\mu > \alpha(2^n.3)'$ and $\mu \neq \alpha(2^N)'$ for any N, then by Lemma 3.3, the first block of 2^{n+1} elements of μ is equal to $\alpha(2^n)$. Therefore, if this holds for every n, $\mu = \lambda$.

The admissible kneading invariant λ is not eventually periodic. For if $|\lambda_k + \lambda_{k+1}t + \lambda_{k+2}t^2 + \ldots|$ were periodic with period n, say, then by (2) the periodic admissible kneading invariant

$$\mu = \inf_m |\lambda_{k+m} + \lambda_{k+m+1}t + \ldots|$$

would be not less than λ. Therefore, by Theorem 3.4 and Proposition 4.3, μ is periodic of period 2^m for some m. Choose an integer N so that $2^N \geqslant k$, and so that $N > m+1$. Then λ and $\alpha = \alpha(2^N)$ have their first 2^{N+1} terms in common, and because $\lambda_i = \lambda_{i+2^N-1}$ for $i \geqslant 2^N$, therefore also $\alpha_i = \alpha_{i+2^N-1}$ for $2^N \leqslant i \leqslant 2^{N+1} - 2^{N-1}$. In particular, the third and fourth blocks of length 2^{N-1} in $\alpha(2^N)$ would have to be equal. But these blocks are $\overline{\alpha(2^{N-2})}$ and $\alpha(2^{N-2})$ respectively, which are certainly not equal.

THEOREM D. *If $\nu(f) < \alpha(2^N)'$ for all integers N, then f has at least one orbit which is not asymptotic to a periodic orbit.*

Proof. For if $\nu(f) < \alpha(2^N)'$ for all integers N, then $\lambda \geqslant \nu(f)$, by Proposition 4.3. Thus by Propositions 1.2 and 1.3 there is a point $x \in I$ such that either $\theta(x) = \lambda$ and 0 is not in the orbit of x, or else $\lambda = \theta(x) \pm t^n\nu(f)$ and $f^n(x) = 0$. In the former case, as a consequence of Lemma 4.0, x is not asymptotically periodic, for λ is not eventually periodic. In the latter case, $\nu(f) = \pm(\lambda_n + \lambda_{n+1}t + \ldots)$, and so $\nu(f)$ is not eventually periodic. In particular, by Lemma 2.2, 0 is not a periodic point. Thus θ is continuous at $f(0)$. Also, from the definition of $\theta(y)$ it is easy to see that $\theta(y) = \theta_0(y) + \theta_0(y)t\theta(f(y))$. That is, if $y \neq 0$, $\theta(f(y)) = (\theta(y) - \theta_0(y))/t\theta_0(y)$. Therefore,

$$\theta(f(0)) = \lim_{y \downarrow 0} \theta(f(y)) = \lim_{y \downarrow 0} \frac{1}{t\theta_0(y)}(\theta(y) - \theta_0(y))$$
$$= t^{-1}(\nu(f) - 1).$$

Thus $\theta(f(0))$ is not eventually periodic. Therefore, by Lemma 4.0, $f(0)$ is not asymptotically periodic.

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