Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.

2. Let $X = \{a, b, c\}$. Which of the following functions define a metric on X?

   (a)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

   (b)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

3. Which of the following functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the triangle inequality? Which are metrics?

   (a) $d(x, y) = \dfrac{|y - x|}{2}$

   (b) $d(x, y) = |x - y| + 1$

   (c) $d(x, y) = \min(|y - x|, \pi)$

   (d) $d(x, y) = 1$

   (e) $d(x, y) = |\log(y/x)|$ (for $x, y > 0$)

4. Consider the set $\mathbb{Z}$ with the Euclidean metric (defined by viewing $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$). What is the ball $B_3(1)$ as a subset of $\mathbb{Z}$? What is the ball $B_{\frac{1}{2}}(1)$?

5. Let (X, d) be a metric space, $r > 0$, and $x \in X$. Show that $x \in B_r(x)$. Conclude in particular that open balls are always non-empty.

6. Let (X, d) be a metric space, and suppose that $r, R \in \mathbb{R}$ satisfy $0 < r \leq R$. Show the containment of the subsets $B_r(x) \subseteq B_R(x)$ of X for any point $x \in X$.

7. Let (X, d) be a metric space, and $r > 0$. For $x, y \in X$, show that $y \in B_r(x)$ if and only if $x \in B_r(y)$.

8. Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$. Let's consider the definititon of an open ball in X,
   $$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}.$$
   Note that the open ball (by definition) consists entirely of elements of X, it is always a subset of X. Let $Y \subseteq X$ be a susbset of X. We showed on our worksheet that Y inherits its own metric structure from tthe metric on X.

   (a) Suppose we are working with both metric spaces X and Y. Given a point $y_0 \in Y$, we can also view y_0 as a point in X. For $r > 0$, let's write
   $$B_r^Y(y_0) = \{y \in Y \mid d(y, y_0) < r\}$$
   for the open ball around y_0 in the metric space Y, and write
   $$B_r^X(y_0) = \{x \in X \mid d(x, y_0) < r\}$$
   for the open ball around y_0 in tthe metric space X. Explain why $B_r^X(y_0)$ and $B_r^Y(y_0)$ could be different sets.

   (b) Show that $B_r^Y(y_0) = B_r^X(y_0) \cap Y$.

   (c) Describe the ball of radius 2 centered around the point $y_0 = 0$ in the metric space Y, where Y is the subset of the real numbers (with the Euclidean metric)

- $Y = \mathbb{R}$
- $Y = [-3, 3]$
- $Y = \left[0, \frac{1}{2}\right]$
- $Y = [0, \infty)$
- $Y = \mathbb{Q}$
- $Y = \mathbb{Z}$

9. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.
   (a) Let x and $r > 0$ be real numbers. Show that $B_r(x)$ is an open interval in $\mathbb{R}$. What are its endpoints?
   (b) Show that every interval of the real line the form (a, b), $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$ is open, for any $a < b \in \mathbb{R}$.
   (c) Show that the interval $[0, 1] \subseteq \mathbb{R}$ is closed.

10. Let (X, d) be a metric space, and let $U \subseteq X$ be a subset. Does the set U necessarily need to be either open or closed? Can it be neither? Can it be both?

11. Let $f : X \to Y$ be a function between sets X and Y. Given a set $A \subseteq X$, recall that its *image* under f is the subset of Y
$$f(A) = \{f(a) \mid a \in A\} \subseteq Y.$$
Given a set $B \subseteq Y$, recall that its *preimage* is the subset of X
$$f^{-1}(B) = \{x \mid f(x) \in B\} \subseteq X.$$
Note that this definition makes sense (and we use the notation $f^{-1}(B)$) even if the function f is not invertible.
   (a) Let $A \subseteq X$. Show that $y \in f(A)$ if and only if there is some $a \in A$ such that $f(a) = y$.
   (b) Let $B \subseteq Y$. Show that $x \in f^{-1}(B)$ if and only if $f(x) \in B$.
   (c) Suppose that $A \subseteq A' \subseteq X$. Show that $f(A) \subseteq f(A')$
   (d) Suppose that $B \subseteq B' \subseteq Y$. Show that $f^{-1}(B) \subseteq f^{-1}(B')$.

12. Let $f, g : X \to \mathbb{R}$ be any functions. What is the relationship between
$$\sup_{x\in\mathbb{R}} f(x) + \sup_{x\in\mathbb{R}} g(x) \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} \Big(f(x) + g(x)\Big) \ ?$$
Show by example that these values need not be equal.

Worksheet problems

(Hand these questions in!)

- Worksheet #1 Problem 1(a), 1(b)

Assignment questions

(Hand these questions in!)

1. Let X be a non-empty set. A function $f : X \to \mathbb{R}$ is called *bounded* if there is some number $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in X$. Let $\mathcal{B}(X, \mathbb{R})$ denote the set of bounded functions from X to $\mathbb{R}$.

(a) Show that the function

$$\begin{aligned} d_\infty : \mathcal{B}(X,\mathbb{R}) \times \mathcal{B}(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ d_\infty(f,g) &= \sup_{x \in X} |f(x) - g(x)| \end{aligned}$$

is well-defined, that is, the suprema always exist.

(b) Show that the function d_∞ defines a metric on $\mathcal{B}(X,\mathbb{R})$.

(c) Explain why the following metric on $\mathbb{R}^n$ is a special case of this construction.

$$\begin{aligned} d_\infty : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ d(\overline{x},\overline{y}) &= \max_{1 \leq i \leq n} |x_i - y_i| \end{aligned}$$

where $\overline{x} = (x_1, \dots, x_n)$ and $\overline{y} = (y_1, \dots, y_n)$.

2. Let (X,d) be a metric space. Show that a nonempty subset $U \subseteq X$ is open if and only if U can be written as a union of open balls in X.

3. Let (X,d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$. Show that the set $\{x \mid d(x_0,x) \leq r\}$ is closed.

4. (a) Prove *DeMorgan's Laws*: Let X be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets of X.

$$(i) \quad X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad\qquad (ii) \quad X \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

Hint: Remember that a good way to prove two sets B and C are equal is to prove that $B \subseteq C$ and that $C \subseteq B$!

(b) Let (X,d) be a metric space, and let $\{C_i\}_{i \in I}$ be a collection of closed sets in X. Note that I need not be finite, or countable! Prove that $\bigcap_{i \in I} C_i$ is a closed subset of X.

(c) Now let (X,d) be a metric space, and let $\{C_i\}_{i \in I}$ be a **finite** collection ($I = \{1, 2, \dots, n\}$) of closed sets in X. Prove that $\bigcup_{i \in I} C_i$ is a closed subset of X.

5. Let (X,d) be a metric space, and consider $Y \subseteq X$ as a metric space under the restriction of the metric to Y. Show by example that a subset $U \subseteq Y$ that is open in Y may or may not be open in X. For each example you should clearly define the sets X, Y, U, and the metric being used, but you may state without proof whether the set U is open in Y and whether it is open in X.