Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X be a topological space with the cofinite topology, and let $S \subseteq X$. Show that the subspace topology on S is the cofinite topology.

2. Let X be a set, and $A \subseteq X$ a proper subset. What are the interior and closure (Assignment Problems 1 and 2) of A if X is given

   (a) the discrete topology? (b) the indiscrete topology?

3. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the topology on X

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

   Find the interior, closures, boundaries, and accumulation points of of the subsets

   (a) $\{a, b, c\}$ (b) $\{a, c, d\}$ (c) $\{a, b, d\}$ (d) $\{b\}$ (e) $\{d\}$ (f) $\{b, d\}$

4. Consider the following subsets of $\mathbb{R}$

   - $\mathbb{R}$
   - $\varnothing$
   - $\{0, 1\}$
   - $[0, \infty)$
   - $(1, 2)$
   - $[1, 2] \cup [3, \infty)$
   - $(-\infty, 0)$
   - $(-\infty, 0]$
   - $\mathbb{N}$
   - $\{-n \mid n \in \mathbb{N}\}$

   Find the interior, closures, boundaries, and accumulation points of these subsets ...

   (a) ... when $\mathbb{R}$ has the discrete topology.

   (b) ... when $\mathbb{R}$ has the indiscrete topology.

   (c) ... when $\mathbb{R}$ has the Euclidean topology.

   (d) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

   (e) ... when $\mathbb{R}$ has the cofinite topology.

   (f) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$. .

   (g) ... when $\mathbb{R}$ has the topology $\mathcal{T} = \{\mathbb{R}\} \cup \{U \subseteq \mathbb{R} \mid 0 \notin U\}$.

Worksheet problems

(Hand these questions in!)

- Worksheet #10 Problems 1, 4.

Assignment questions

(Hand these questions in!)

In this assignment, we will generalize the notions of interior, closure, boundary, and accumulation point to general topological spaces—and verify that many of the results we proved for metric spaces hold in general. This is an opportunity to revisit their proofs!

1. **Definition (Interior of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. A point $a \in A$ is an *interior point* of A if there exists an open subset $U \subseteq X$ such that $a \in U \subseteq A$. Define the *interior* of A to be the set of interior points of A. Concretely,

$$\mathrm{Int}(A) = \{ \, a \in A \mid \text{there is some open neighbourhood } U \text{ of } a \text{ such that } U \subseteq A \, \}.$$

Observe that, by definition, $\mathrm{Int}(A) \subseteq A$. Prove the following.

(a) A is open if and only if $A = \mathrm{Int}(A)$.

(b) $\mathrm{Int}(A)$ is an open set.

(c) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

(d) Suppose that $A \subseteq X$ is any subset, and $U \subseteq A$ is an open subset of X. Prove that $U \subseteq \mathrm{Int}(A)$.

(e) $\mathrm{Int}(A) = \displaystyle\bigcup_{\substack{U \subseteq A, \\ U \text{ open in } X}} U.$

2. **Definition (Closure of a set in a topological space).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Define the *closure* of A to be the set

$$\overline{A} = \{ \, x \in X \mid \text{any neighbourhood } U \text{ of } x \text{ contains a point of } A \, \}.$$

Prove the following.

(a) $A \subseteq \overline{A}$.

(b) A is closed if and only if $A = \overline{A}$.

(c) $\overline{\overline{A}} = \overline{A}$.

(d) $\overline{A}$ is a closed set.

(e) Suppose that $A \subseteq X$ is any subset, and C is a closed set containing A. Then $\overline{A} \subseteq C$.

(f) $\overline{A} = \displaystyle\bigcap_{\substack{C \text{ closed in } X, \\ A \subseteq C}} C.$

(g) **Definition (Accumulation points of a set).** Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open neighbourhood U of x also contains a point in S distinct from x.

By Homework #2 Problem 4(b), when the topology on X is induced by a metric, this agrees with our metric space definition of an accumulation point.

Let A' be the set of accumulation points of A. Then $\overline{A} = A \cup A'$.

3. **Definition (Boundary of a set A).** Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \text{Int}(A)$.

(a) Prove that $\partial A = \overline{A} \cap \overline{(X \setminus A)}$.

(b) Use this result to conclude that (*i*) ∂A is closed, and (*ii*) $\partial A = \partial(X \setminus A)$.

(c) Prove the following.

Theorem (An equivalent definition of ∂A). Let X be a topological space, and let $A \subseteq X$. Then

$$\partial A = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{every open neighbourhood } U \text{ of } x \text{ contains at least one point of } A, \\ \text{and at least one point of } X \setminus A \end{array} \right\}$$

(d) Prove that every point of X falls into one of the following three categories of points, and that the three categories are mutually exclusive:

(i) interior points of A;

(ii) interior points of $X \setminus A$;

(iii) points in the (common) boundary of A and $X \setminus A$.

4. **Definition (Topological equivalence).** Let X be a set. Let d_1 and d_2 be two metrics on X. Then the metric spacess (X, d_1) and (X, d_2) are called *topologically equivalent* if the metrics induce the same topology on X.

(a) Let X be a nonempty finite set. Show that all metrics on X are topologically equivalent. *Hint:* Homework #8 Problem 2.

(b) Consider the natural numbers $\mathbb{N}$. Show that ($\mathbb{N}$, Euclidean) and ($\mathbb{N}$, discrete) are topologically equivalent. Conclude that a bounded metric space strucure can be topologically equivalent to an unbounded metric space structure.

(c) Prove the following theorem.

Theorem (A criterion for topological equivalence). Let X be a set and let d_1 and d_2 be two metrics on X. For each metric d_i, write $B_r^{d_i}(x)$ to denote the ball of radius r centred on x defined with respect to the metric d_i. Show that d_1 and d_2 are topologically equivalent if and only if the following condition holds: For every $\epsilon > 0$, there exists $r_1, r_2 > 0$ such that

$$B_{r_1}^{d_1}(x) \subseteq B_\epsilon^{d_2}(x) \qquad \text{and} \qquad B_{r_2}^{d_2}(x) \subseteq B_\epsilon^{d_1}(x).$$

(d) Let (X, d) be a topological space. Show that the following metric $\overline{d}$ on X is topologically equivalent to d. (You do not need to verify that it is a metric).

$$\overline{d}(x, y) = \begin{cases} d(x, y), & \text{if } d(x, y) < 1, \\ 1, & \text{if } d(x, y) \geq 1. \end{cases}$$

This result shows that every metric is topologically equivalent to a bounded metric. In particular, whenever a topological space is metrizable, we can always choose the corresponding metric to be bounded.

(e) Let X be a set. Show that two metric space structures (X, d_1) and (X, d_2) on X are topologically equivalent if and only if the map

$$\begin{aligned} I : (X, d_1) &\longrightarrow (X, d_2) \\ x &\longmapsto x \end{aligned}$$

is a homeomorphism. *Hint:* Homework #4 Problem 2(b).