Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let (X, d) be a metric space. Verify that the collection $\mathcal{T}_d$ of open sets in this metric space does indeed form a topology on the set X.

2. Find all possible topologies on the set $X = \{0, 1\}$.

3. Let $X = \{0, 1, 2\}$. Show that the collection of subsets $\Big\{\varnothing, X, \{0, 1\}, \{1, 2\}\Big\}$ is **not** a topology on X.

4. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show by induction that the intersection of any **finite** collection of open sets is open.
   (b) Explain why this argument does not apply to an infinite collection of open sets.

5. Consider the following collection of subsets of $\mathbb{R}$. Note that $\varnothing$ is considered a finite set.

$$\mathcal{R} = \{A \mid A \text{ is finite}\} \cup \{\mathbb{R}\}.$$

   (a) Show that $\varnothing, \mathbb{R} \in \mathcal{R}$.
   (b) Show that if $U, V \in \mathcal{R}$, then so is $U \cap V$.
   (c) Show that if $U, V \in \mathcal{R}$, then so is $U \cup V$. Show, in fact, that the union of any finite number of elements in $\mathcal{R}$ is an element of $\mathcal{R}$.
   (d) Show that arbitrary unions of elements in $\mathcal{R}$ may not be in $\mathcal{R}$. Conclude that $\mathcal{R}$ is not a topology on $\mathbb{R}$.
   (e) Conclude that the second axiom of a topology—that it is closed under arbitrary unions—is truly stronger than the condition that it be closed under finite unions.

6. Let X be a set. Show that the discrete topology on X is induced by the discrete metric on X.

7. Let X be a set. Show that if X is a finite set, then the cofinite topology (defined in Assignment Problem 3) coincides with the discrete topology on X.

8. Let X be a topological space consisting of either 0 or 1 points. Explain why X satisfies both the T_1 and T_2 properties.

Worksheet problems

(Hand these questions in!)

- Worksheet #9 Problems 3, 4.

Assignment questions

(Hand these questions in!)

1. (a) **Definition (T_1-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that X has the T_1 *property*, or call X a T_1*-space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists some neighbourhood U_x of x that does not contain y, and there exists some neighbourhood U_y of y that does not contain x.

   Prove that $(X, \mathcal{T})$ is a T_1-space if and only if, for every point $x \in X$, the singleton set $\{x\}$ is closed. (Mathematicians often refer to this property by the slogan "points are closed").

   (b) **Definition (Hausdorff property; T_2-space).** Let $(X, \mathcal{T})$ be a topological space. Then we say that X has the T_2 *property*, or call X a *Hausdorff space*, if it satisfies the following condition: For every pair of distinct points $x, y \in X$, there exists open neighbourhoods U_x of x and U_y of y such that U_x and U_y are disjoint.

   Show that every space satisfying the T_2 property must satisfy the T_1 property. Conclude in particular that, in a Hausdorff space, singleton sets $\{x\}$ are closed.

2. Let $(X, \mathcal{T})$ be a **finite** topological space. Show that, if X has the T_1 property, then the topology on X must be the discrete topology.

3. (a) Let X be a nonempty set, and let $\mathcal{T}$ be the collection of subsets

   $$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X \mid X \setminus U \text{ is a finite set}\}.$$

   Verify that $\mathcal{T}$ is a topology on X. It is called the *cofinite* topology (a contraction of "complements are finite").

   (b) Consider the set $\mathbb{R}$ with the cofinite topology. Verify that $(\mathbb{R}, \text{cofinite})$ is a T_1-space but not Hausdorff. This shows that the T_2 condition is strictly stronger than the T_1 condition.

   (c) Conclude that the cofinite topology on $\mathbb{R}$ is not metrizable. *Hint:* Worksheet #4 Problem 2(a).