Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an example of a metric space (X, d) and a continuous function $f : X \to \mathbb{R}$ such that f has a finite supremum on X, but f does not achieve its supremum at any point $x \in X$.

2. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.

3. Let (X, d_X) and (Y, d_Y) be bounded metric spaces. Show that $(X \times Y, d_{X\times Y})$ is bounded. What if we only assumed X is bounded?

4. Let (X, d_X) and (Y, d_Y) be metric spaces, and consider their product metric $(X \times Y, d_{X\times Y})$. Prove that a subset $U \subseteq X \times Y$ is open if and only if U can be written as the union of subsets of the form $U_x \times U_y$, with U_x an open subset of X and U_y an open subset of Y.

5. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$ and $B \subseteq Y$. Then A and B inherit metric space structures from the metrics on X and Y. Consider the Cartesian product $A \times B$. There are two ways we can define a metric on $A \times B$: (1) using the product metric defined by the metric spaces A and B, and (2) as a subspace of the metric space $X \times Y$ with the product metric $d_{X\times Y}$. Verify that these two metrics agree, so there is no ambiguity in how to metricize the product $A \times B$.

Worksheet Problems

(Hand these questions in!)

- Worksheet #7 Problems 1, 2, 3.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a continuous function of metric spaces. For each of the following statements, either prove the statement, or state a counterexample. You may state the counterexample without proof.
   (a) If $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.
   (b) If $C \subseteq X$ is closed, then $f(C) \subseteq Y$ is closed.
   (c) If $C \subseteq X$ is sequentially compact, then $f(C) \subseteq Y$ is sequentially compact.
   (d) If $C \subseteq Y$ is sequentially compact, then $f^{-1}(C)$ is sequentially compact.
   (e) If $B \subseteq X$ is bounded, then $f(B) \subseteq Y$ is bounded.
   (f) If $B \subseteq Y$ is bounded, then $f^{-1}(B) \subseteq X$ is bounded.
   (g) If X is complete, then $f(X)$ (with metric space structure inherited from Y) is complete.

Hint: Below are some continuous functions that are useful sources of counterexamples. Subsets of $\mathbb{R}$ have the Euclidean metric unless otherwise noted.

$$\begin{aligned} (\mathbb{R}, \text{discrete}) &\longrightarrow (\mathbb{R}, \text{Euclidean}) \\ x &\longmapsto x \end{aligned} \qquad \begin{aligned} \mathbb{R} &\longrightarrow (-1,1) \\ x &\longmapsto \frac{\arctan(x)}{\pi/2} \end{aligned}$$

$$\begin{aligned} (0,\infty) &\longrightarrow (0,\infty) \\ x &\longmapsto \frac{1}{x} \end{aligned} \qquad \begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 0 \end{aligned}$$

2. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $C \subseteq X$ and $D \subseteq Y$ are closed subsets. Prove or find a counterexample: the subset $C \times D \subseteq X \times Y$ is closed with respect to the product metric.

3. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be a metric spaces, and suppose that $f : Z \to X$ and $g : Z \to Y$ are continuous functions. Prove that the function

$$\begin{aligned} (f \times g) : Z &\longrightarrow X \times Y \\ (f \times g)(z) &= \Big(f(z), g(z)\Big) \end{aligned}$$

is continuous with respect to the product metric $d_{X \times Y}$ on $X \times Y$.

4. Let (X, d_X) and (Y, d_Y) be nonempty metric spaces.
   (a) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of points in X and Y, respectively. Show that the sequence of points $\big((x_n, y_n)\big)_{n\in\mathbb{N}}$ in $(X \times Y, d_{X\times Y})$ converges if and only if both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge.
   Hint: Worksheet # 7 Problems 2(b) and 3(a).
   (b) Show that $X \times Y$ is sequentially compact if and only if both X and Y are.

5. (a) Suppose that B is a nonempty closed and bounded subset of $\mathbb{R}$. Explain why the Least Upper Bound Property of $\mathbb{R}$ implies that the supremum of B exists, and show that B contains its supremum.
   (b) Prove the following theorem. This theorem is one of the important reasons we care about sequential compactness!

   Theorem (Extreme value theorem for metric spaces). Let (X, d) be a metric space and C a nonempty, sequentially compact subset of X. Let $f : X \to \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that

   $$f(c) = \sup_{x \in C} f(x).$$

   In other words, the restriction $f|_C$ achieves its supremum.