## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

- 1. Consider the sequence of real numbers 1, 2, 3, 4, 5, .... Which of the following are subsequences?
  - (c) 2, 4, 6, 8, 10, 12, ... (a)  $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$
  - (d) 1, 3, 2, 4, 5, 7, 6, 8, 9, 11, ... (b) 1, 1, 1, 1, 1, 1, 1, ...
- 2. Let (X,d) be a metric space, and let  $(a_n)_{n\in\mathbb{N}}$  be a sequence of points in X. Recall that we proved that, if  $\lim_{n\to\infty} a_n = a_\infty$ , then any subsequence of  $(a_n)_{n\in\mathbb{N}}$  also converges to  $a_\infty$ .
  - (a) Suppose that  $(a_n)_{n\in\mathbb{N}}$  has a subsequence that does not converge. Prove that  $(a_n)_{n\in\mathbb{N}}$  does not converge.
  - (b) Suppose that  $(a_n)_{n \in \mathbb{N}}$  has a subsequence converging to  $a \in X$ , and a different subsequence converging to  $b \in X$ , with  $a \neq b$ . Prove that  $(a_n)_{n \in \mathbb{N}}$  does not converge.
- 3. Let (X,d) be a metric space, and let  $(a_n)_{n\in\mathbb{N}}$  be a sequence of points in X. Show that, if  $\{a_n \mid n \in \mathbb{N}\}\$  is a finite set, then  $(a_n)_{n \in \mathbb{N}}$  must have a subsequence that is constant (and, in particular, convergent).
- 4. Let (X, d) be a metric space, and let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of points in X. Suppose that the set  $\{a_n \mid n \in \mathbb{N}\}$  is unbounded. Explain why  $(a_n)_{n \in \mathbb{N}}$  cannot converge.
- 5. Find examples of sequences  $(a_n)_{n \in \mathbb{N}}$  of real numbers with the following properties.
  - (a)  $\{a_n \mid n \in \mathbb{N}\}\$  is unbounded, but  $(a_n)_{n \in \mathbb{N}}$  has a convergent subsequence
  - (b)  $(a_n)_{n \in \mathbb{N}}$  has no convergent subsequences
  - (c)  $(a_n)_{n \in \mathbb{N}}$  is not an increasing sequence, but it has an increasing subsequence
  - (d)  $(a_n)_{n \in \mathbb{N}}$  has four subsequences that each converge to a distinct limit point
- 6. Write down a bounded sequence in  $\mathbb{R}$ . Can you identify a convergent subsequence?
- 7. Determine which of the following subsets of  $\mathbb{R}^2$  can be expressed as the Cartesian product of two subsets of  $\mathbb{R}$ .

  - $\begin{array}{ll} \text{(a)} & \{(x,y) \mid x \in \mathbb{Q}\} & \text{(c)} & \{(x,y) \mid x > y\} & \text{(e)} & \{(x,y) \mid x^2 + y^2 < 1\} & \text{(g)} & \{(x,y) \mid x = 3\} \\ \text{(b)} & \{(x,y) \mid x, y \in \mathbb{Q}\} & \text{(d)} & \{(x,y) \mid 0 < y \le 1\} & \text{(f)} & \{(x,y) \mid x^2 + y^2 = 1\} & \text{(h)} & \{(x,y) \mid x + y = 3\} \\ \end{array}$
- 8. (a) For which values of r is the square  $(-r,r) \times (-r,r) \subseteq \mathbb{R}^2$  contained in the unit ball  $\{(x,y) \mid x^2 + y^2 < 1\}$ ?
  - (b) For which values of r is the r-ball  $\{(x, y) \mid x^2 + y^2 < r^2\}$  contained in the square  $(-1, 1) \times$  $(-1,1) \subseteq \mathbb{R}^2$ ?

## Worksheet problems

(Hand these questions in!)

• Worksheet #6, Problem 2

## Assignment questions

(Hand these questions in!)

- 1. For sets X and Y, let  $A, B \subseteq X$  and  $C, D \subseteq Y$ . Consider the Cartesian product  $X \times Y$ . For each of the following, determine whether you can replace the symbol  $\Box$  with  $\subseteq, \supseteq, =$ , or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.
  - (a)  $(A \times C) \cup (B \times D)$   $\Box$   $(A \cup B) \times (C \cup D)$
  - (b)  $(A \times C) \cap (B \times D)$   $\Box$   $(A \cap B) \times (C \cap D)$
  - (c)  $(X \setminus A) \times (Y \setminus C) \quad \Box \quad (X \times Y) \setminus (A \times C)$
- 2. Consider the real numbers  $\mathbb{R}$  with the Euclidean metric. Determine the interior, closure, and boundary of the subset  $\mathbb{Q} \subseteq \mathbb{R}$ . Remember to rigorously justify your solution!
- 3. Let (X, d) be a metric space, and  $(a_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in X. Show that  $\{a_n \mid n \in \mathbb{N}\}$  is a bounded subset of X.
- 4. Recall the following result from real analysis (which you do not need to prove):

**Theorem (Bolzano–Weierstrass).** Let  $A \subseteq \mathbb{R}^n$  be a bounded infinite set. Then A has an accumulation point  $x \in \mathbb{R}^n$ .

Prove the following result (which is sometimes also called the Bolzano–Weierstrass theorem):

**Theorem (Sequential compactness in**  $\mathbb{R}^n$ ). Consider the space  $\mathbb{R}^n$  with the Euclidean metric. Let  $S \subseteq \mathbb{R}^n$  be a subset. Then S is sequentially compact if and only if S is closed and bounded.

5. (a) **Definition (Open cover).** A collection  $\{U_i\}_{i \in I}$  of open subsets of a metric space X is an *open cover* of X if  $X = \bigcup_{i \in I} U_i$ . In other words, every point in X lies in some set  $U_i$ .

Suppose that (X, d) is a sequentially compact metric space. Let  $\mathcal{U} = \{U_i\}_{i \in I}$  be an open cover of X. Prove that (associated to the open cover  $\mathcal{U}$ ) there exists a real number  $\delta > 0$ with the following property: for every  $x \in X$ , there is some associated index  $i_x \in I$  such that  $B_{\delta}(x) \subseteq U_{i_x}$ .

(b) **Definition (** $\epsilon$ **-nets of a metric space).** Let (X, d) be a metric space. A subset  $A \subseteq X$  is called an  $\epsilon$ -net if  $\{B_{\epsilon}(a) \mid a \in A\}$  is an open cover of X.

Suppose that (X, d) is a sequentially compact metric space, and  $\epsilon > 0$ . Prove that X has a finite  $\epsilon$ -net.

(c) Let (X, d) be a sequentially compact metric space, and let  $\mathcal{U} = \{U_i\}_{i \in I}$  be an open cover of X. Show that there exists some finite collection  $U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$  so that  $\{U_{i_1}, \ldots, U_{i_n}\}$ covers X, i.e., so that  $X = U_{i_1} \cup \cdots \cup U_{i_n}$ .

We will return to these results later in the course when we study *compactness*.