Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X, and let $a_\infty \in X$. Show that the following statement is equivalent to the statement that $\lim_{n\to\infty} a_n = a_\infty$.

   For every $\epsilon > 0$, the ball $B_\epsilon(a_\infty)$ about a_∞ contains a_n for all but finitetly many n.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X.
   (a) Let $x \in X$. Negate the definition of convergence to state what it means for the sequence to **not** converge to x.
   (b) Formally state what it means for the sequence $(a_n)_{n\in\mathbb{N}}$ to be non-convergent.

3. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.

   (a) $a_n = 0$ (b) $a_n = \dfrac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$

4. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞, respectively. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $(a_\infty + b_\infty)$.

5. Consider the sequence $\left(\dfrac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Let $\epsilon > 0$ be fixed. Find a number $N \in \mathbb{R}$ so that, for all $m, n \geq N$,
   $$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| < \epsilon.$$
   This shows that the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is *Cauchy* (as defined in Question 5).

Worksheet Problems

(Hand these questions in!)

- Worksheet 4 Problem 1, 2

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f^{-1}(C \cup D) \quad \square \quad f^{-1}(C) \cup f^{-1}(D)$ (b) $f^{-1}(C \cap D) \quad \square \quad f^{-1}(C) \cap f^{-1}(D)$

   (c) For $C \subseteq D$, $\quad f^{-1}(D \setminus C) \quad \square \quad f^{-1}(D) \setminus f^{-1}(C)$

2. **Definition (The discrete metric.)** Given a set X, the *discrete metric on* X is the metric $d_X : X \times X \to \mathbb{R}$ defined by

$$d_X(x, x') = \begin{cases} 0, & x = x' \\ 1, & x \neq x' \end{cases} \qquad \text{for all } x, x' \in X.$$

Let (X, d) be a metric space with the discrete metric.

(a) Show that every subset of X is both open and closed.

(b) Let (Y, d_Y) be any metric space. Prove that **every** function $f : X \to Y$ is continuous.

3. In this question, we will prove the following result.

Theorem (Another characterization of closed subsets). Let (X, d) be a metric space, and let $A \subseteq X$. Then A is closed if and only if it satisfies the following condition: If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence of points in A converging to a point $a_\infty \in X$, then the limit a_∞ is contained in A.

(a) Suppose that $A \subseteq X$ is closed. Let a_∞ be the limit of a convergent sequence $(a_n)_{n\in\mathbb{N}}$ of points in A. Show that $a_\infty \in A$.

(b) Suppose that $A \subseteq X$ is a subset that contains the limits of every one of its convergent sequences. Prove that A is closed.

4. Prove the following result:

Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n\in\mathbb{N}}$ in X, then $\big(f(a_n)\big)_{n\in\mathbb{N}}$ converges in Y, and

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

5. Consider the following definition.

Definition (Cauchy sequence.) Let (X, d) be a metric space. Then a sequence $(a_n)_{n\in\mathbb{N}}$ of points in X is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists some $N \in \mathbb{R}$ such that $d(a_n, a_m) < \epsilon$ whenever $n, m \geq N$.

(a) Prove that every convergent sequence in X is a Cauchy sequence.

(b) Give an example of a metric space (X, d) and a sequence $(a_n)_{n\in\mathbb{N}}$ in X that is Cauchy but does not converge. Fully justify your solution!