Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

2. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   Is X connected?

3. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem b.
   (b) Show that every interval must have one of these forms.

4. Give an example of a subset A of $\mathbb{R}$ (with the standard topology) such that A is not connected, but $\overline{A}$ is connected. (Compare to Assignment Problem **??**)

5. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition that X is compact is stronger than the assumption that X has a finite open cover.
   (b) Show that every topological space has a finite open cover.
   Hint: What is the first axiom of a topology?

6. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset A is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset A is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^{n} U_i$.

7. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) A is compact, and B is noncompact
   (b) B is compact, and A is noncompact

8. Determine the connected components of $\mathbb{R}$ with the following topologies.
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

Assignment questions

(Hand these questions in!)

1. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a continuous map. Prove that if X is connected, then $f(X)$ is connected. In other words, the continuous image of a connected space is connected.

   (b) Recall the Intermediate Value Theorem from real analysis (which you may use without proof).

   > **Intermediate Value Theorem.** If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

   Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ and z lies between x and y, then $z \in A$.
   Prove that any interval of $\mathbb{R}$ is connected. *Hint:* Worksheet #15 Problem 4.

   (c) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

   These last two results together prove:

   > **Theorem (Connected subsets of $\mathbb{R}$).** A subset of $\mathbb{R}$ is a connected if and only if it is an interval.

2. (a) Prove the following result.

   > **Theorem (Generalized Intermediate Value Theorem).** Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

   (b) Prove that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (In other words, show that there is some $x \in [0, 1]$ so that $f(x) = x$).
   Hint: Consider the function

   $$\begin{aligned} g &: [0,1] \to \mathbb{R} \\ g(x) &= f(x) - x. \end{aligned}$$

3. Prove the following result. This theorem is a major reason we care about compactness!

   > **Theorem (Generalized Extreme Value Theorem).** Let X be a nonempty compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, f achieves its supremum on X.

   Hint: See Worksheet #17 Problem 2, 4(a), 4(b), and Homework #6 Problem 5(a).

4. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

(b) Prove that any compact metric space is sequentially compact.

Combined with Homework #5 Problem 5, this exercise proves:

> **Theorem (Compactness vs sequential compactness in metric spaces).** Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

(Neither direction of this theorem holds, however, for arbitrary topological spaces!)

Combined with Homework #5 Problem 4, this exercise proves:

> **Theorem (Compactness in $\mathbb{R}^n$).** Endow $\mathbb{R}^n$ with the Euclidean metric. A subspace $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.