Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   (a) Is X a T_1-space?
   (b) Is X Hausdorff?
   (c) Find the interior, closure, and boundary of $\{a, c, d\}$.
   (d) Find the interior, closure, and boundary of $\{a, b\}$.

2. (a) Let $(X, \mathcal{T})$ be a topological space, and let $I : X \to X$ be the identity function, defined by $I(x) = x$ for all $x \in X$. Show that I is always continuous when the domain and codomain are given the same topology.
   (b) Now consider $I : \mathbb{R} \to \mathbb{R}$, and investigate whether I is continuous when we allow the domain and codomain to carry different topologies – the answer now depends on the two topologies chosen. Consider (for example) the discrete topology, the indiscrete topology, the cofinite topology, and the Euclidean topology.

3. Let $(X, \mathcal{T})$ be a topological space.
   (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
   (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.

4. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).

5. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.
   (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
   (b) Show that if f and g are both injective, then $g \circ f$ is injective.
   (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
   (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.

Worksheet problems

(Hand these questions in!)

- Worksheet #11 Problems 2(a), 3.
- Worksheet #12 Problems 1, 4, 5.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$.
   (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (b) Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.

2. Let $(X, \mathcal{T}_X)$ be a topological space, and suppose $X = A \cup B$ for two **closed** subsets $A, B \subseteq X$. Let $(Y, \mathcal{T}_Y)$ be a topological space and $f : X \to Y$ a function. Show that, if $f|_A$ and $f|_B$ are continuous (with respect to the subspace topologies on A and B), then f is continuous.

3. (a) Suppose that $(X, \mathcal{T}_X)$ is a topological space, and that $(Y, \mathcal{T}_Y)$ is a **Hausdorff** topological space. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that
   $$f(a) = g(a) \qquad \text{for all } a \in A.$$
   Prove that
   $$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$
   This says that the values of a continuous function on $\overline{A}$ are completely determined by its values on A.
   (b) Suppose you have a function of topological spaces $f : \mathbb{R} \to Y$, where $\mathbb{R}$ has the standard topology and Y is Hausdorff. Briefly explain why the previous problem implies, if f is continuous, then it is completely determined by its values on $\mathbb{Q}$.

4. (a) Show that the continuous image of a Hausdorff topological space need not be Hausdorff. Specifically, find topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ such that X is Hausdorff but $f(X)$ (as a subspace of Y with the subspace topology) is not Hausdorff.
   (b) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and assume that $(Y, \mathcal{T}_Y)$ is Hausdorff. Show that, if there exists a continuous **injective** map $f : X \to Y$, then X must also be Hausdorff.

5. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. For a subset $A \subseteq X$, prove the following.
   (a) $\text{Int}(A) = \{a \in A \mid \text{there exists a basis element } B \text{ with } a \in B \subseteq A\}$
   (b) $\overline{A} = \{x \in X \mid \text{any basis element } B \in \mathcal{B} \text{ containing } x \text{ must contain a point of } A\}$