The core entropy of quadratic maps and the infinite clique polynomial

Giulio Tiozzo
Yale University

In honor of J.H.Hubbard
Bremen, August 18, 2015

Summary

1. Topological entropy

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2. Core entropy

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3. Thurston's algorithm

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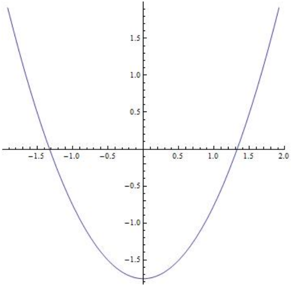

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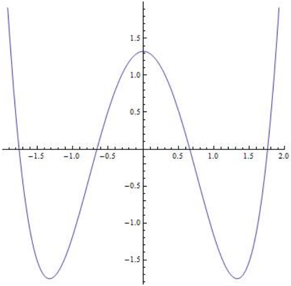

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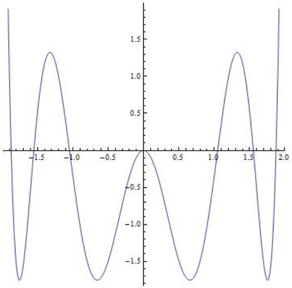

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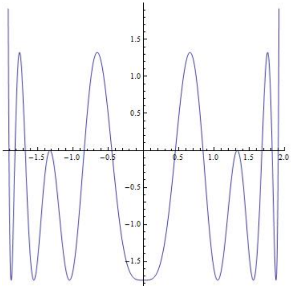

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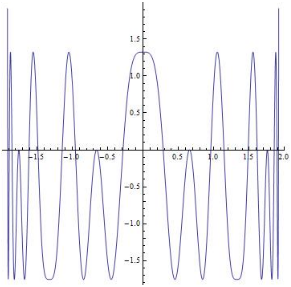

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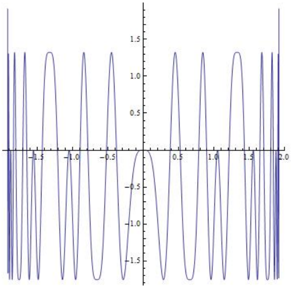

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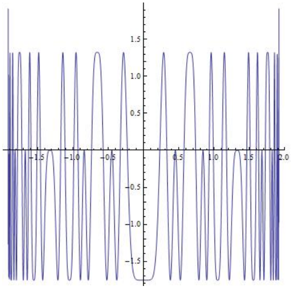

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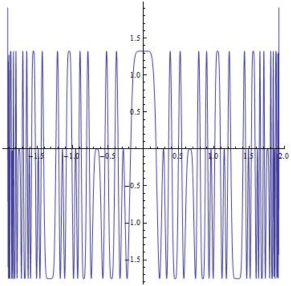

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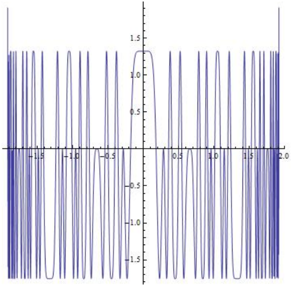
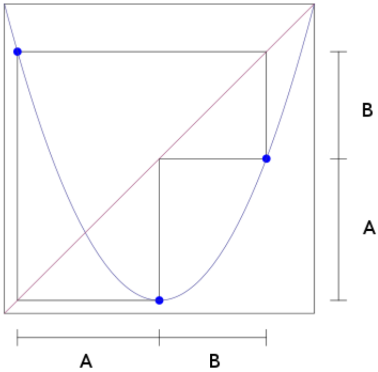


Entropy of postcritically finite maps (the airplane)

Entropy is the logarithm of the leading eigenvalue of the transition matrix for a Markov partition.



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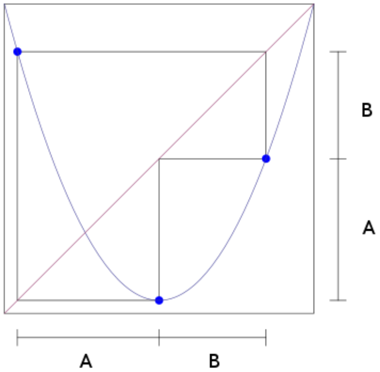

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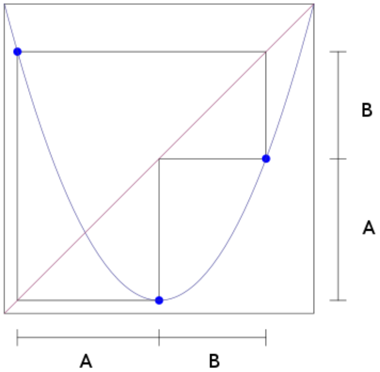

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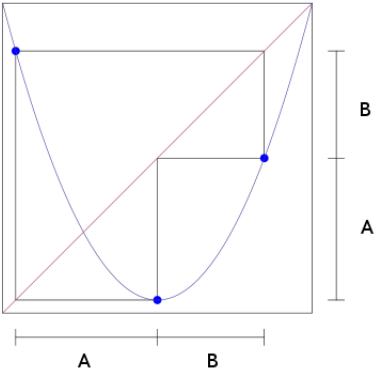

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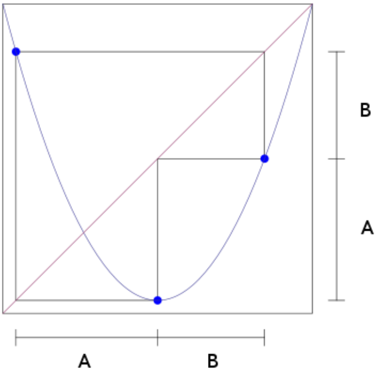

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How does entropy change with the parameter c?

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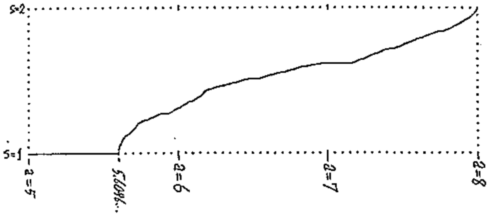
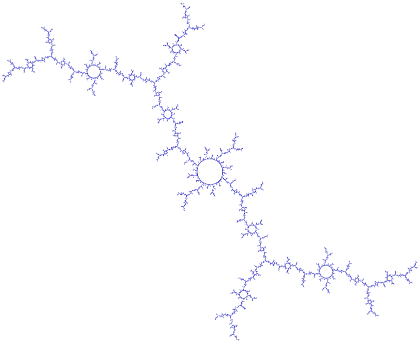


Remark. If we consider $f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ entropy is constant $h_{top}(f_c, \hat{\mathbb{C}}) = \log 2$. (Lyubich 1980)

The complex case: Hubbard trees

The Hubbard tree T_c of a quadratic polynomial is a forward invariant, connected subset of the filled Julia set which contains the critical orbit.



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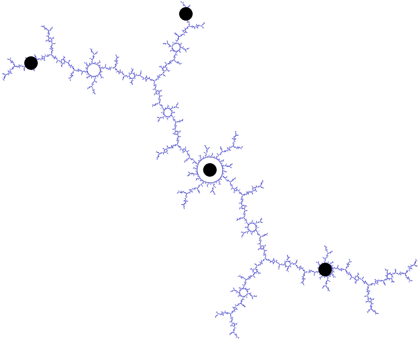

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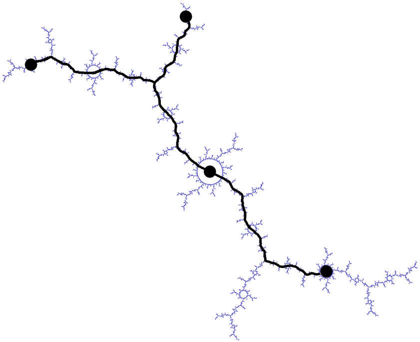

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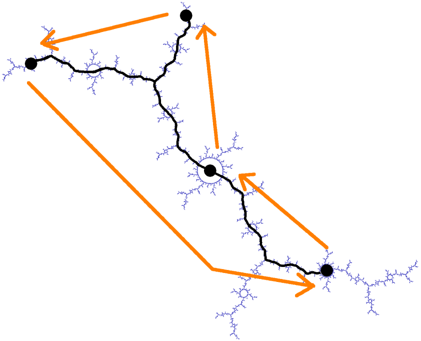
The Hubbard tree T_c of a quadratic polynomial is a forward invariant, connected subset of the filled Julia set which contains the critical orbit. The map f_c acts on it.



The core entropy

Let $\theta \in \mathbb{Q}/\mathbb{Z}$. Then the external ray at angle θ lands, and determines a postcritically finite quadratic polynomial f_θ, with Hubbard tree T_θ.

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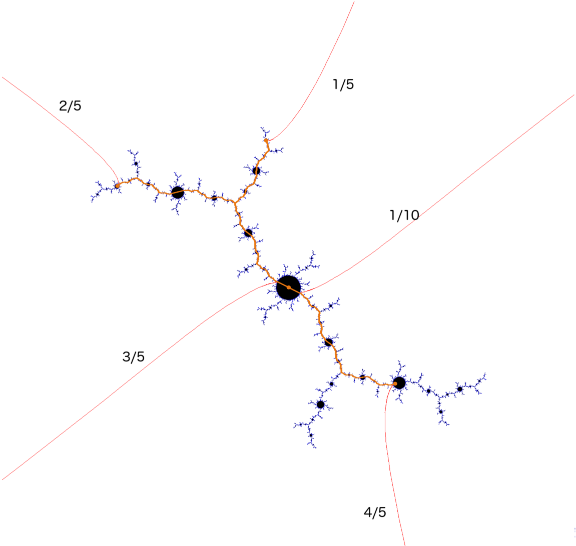

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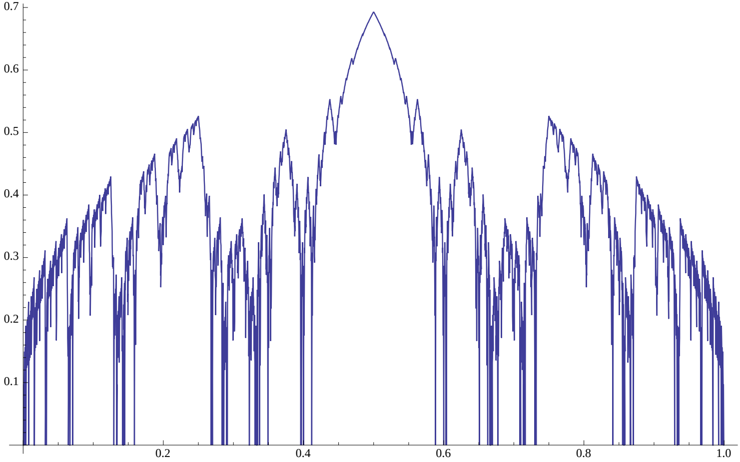
Question: How does $h(\theta)$ vary with the parameter θ?

Core entropy as a function of external angle (W. Thurston)



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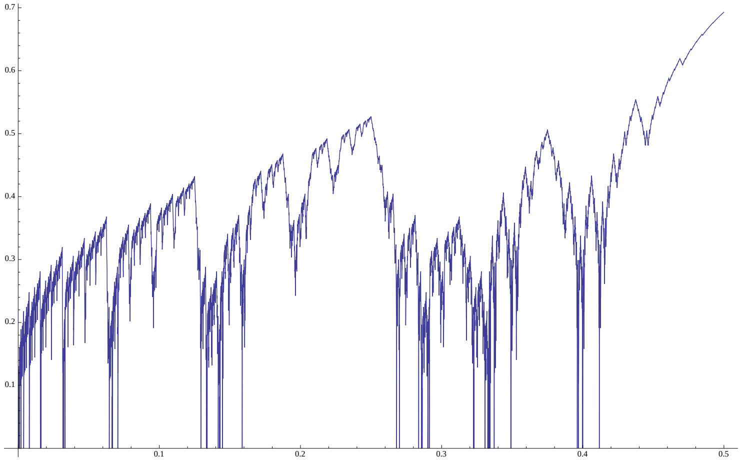

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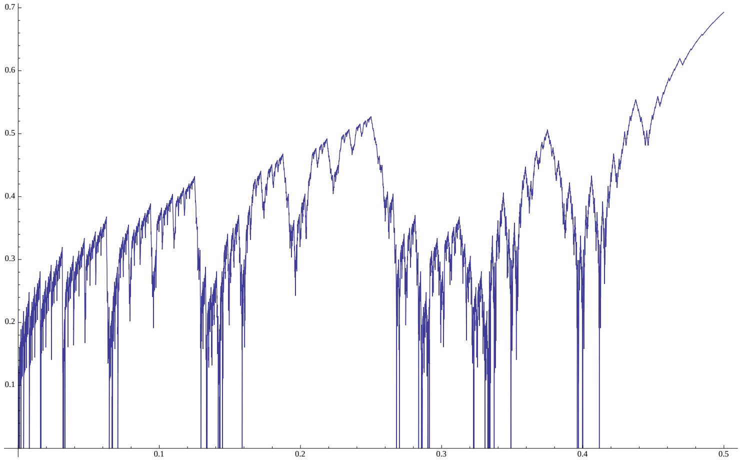

Question Can you see the Mandelbrot set in this picture?

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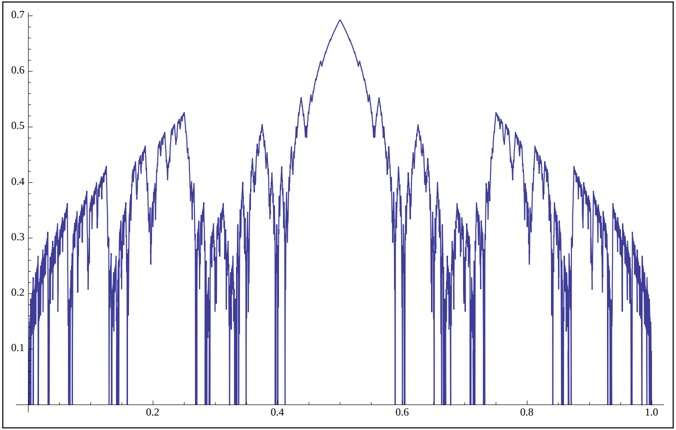
- Core entropy also proportional to Hausdorff dimension of angles landing on the corresponding vein (T.)

The core entropy as a function of external angle

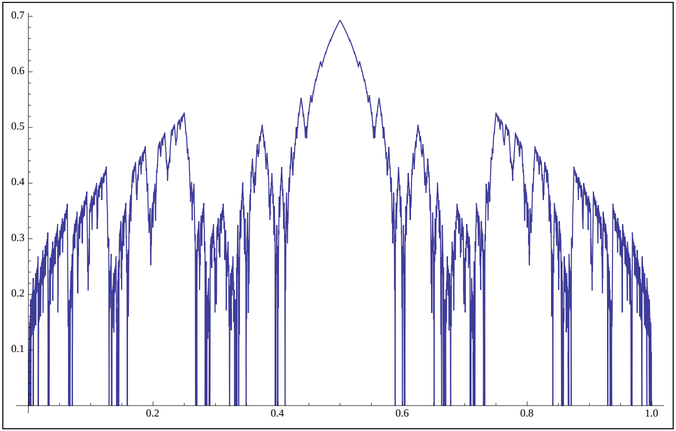
Question (Thurston, Hubbard):
Is $h(\theta)$ a continuous function of θ?



The Main Theorem: Continuity

Theorem

The core entropy function $h(\theta)$ extends to a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.



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Idea 2: (Thurston): look at set of pairs of postcritical points, which correspond to arcs between postcritical points.
Denote $c_i := f^i(0)$ the i^{th} iterate of the critical point, and let

$$P := \{(c_i, c_j) \ \ i, j \geq 0\}$$

the set of pairs of postcritical points

Computing the entropy: non-separated pair

A pair (i, j) is non-separated if c_i and c_j lie on the same side of the critical point.

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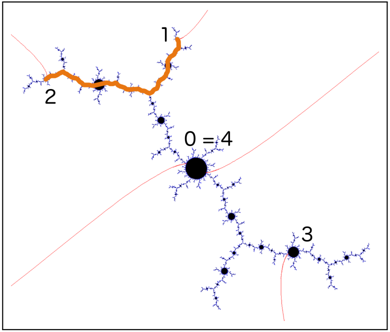

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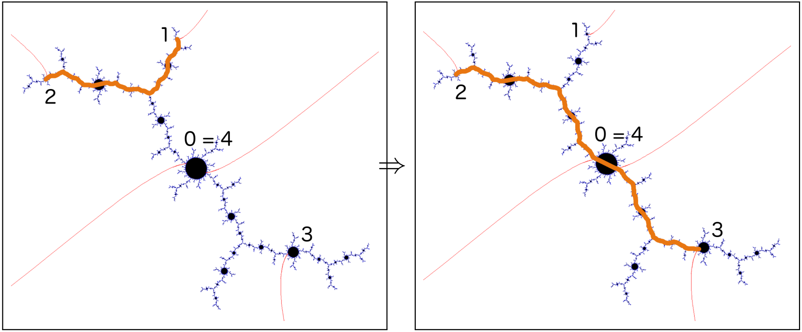

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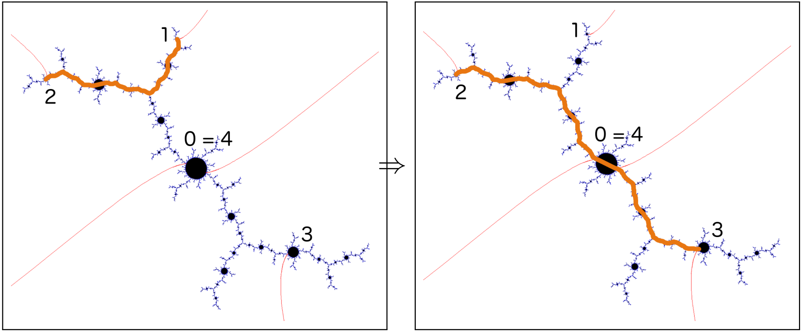

$(1, 2) \quad \Rightarrow \quad (2, 3)$

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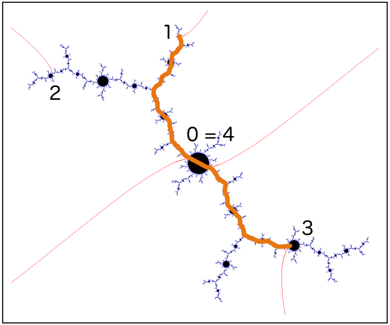

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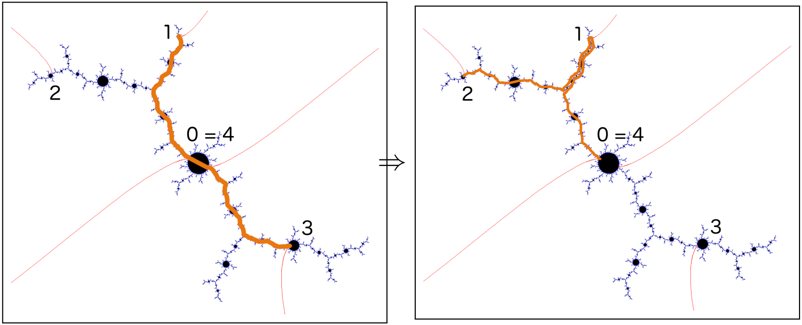

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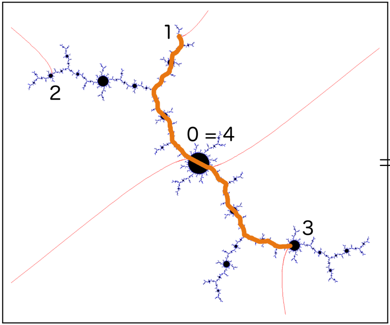
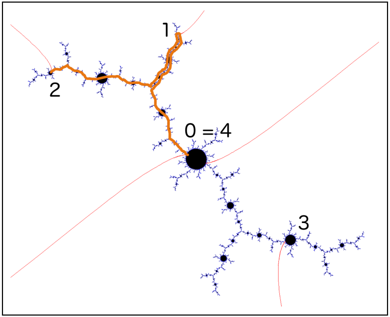


$\Rightarrow$



$$(1, 3) \quad \Rightarrow \quad (1, 2) + (1, 4)$$

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- If (i,j) is separated, then $(i,j) \to (1, i+1) + (1, j+1)$.

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Theorem (Thurston; Tan Lei)

The entropy of f_θ is given by

$$h(\theta) = \log \lambda$$

where λ is the leading eigenvalue of A.

See also Gao, Jung.

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$$P(t) := \det(I - tA)$$

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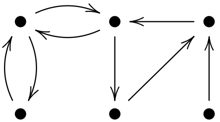

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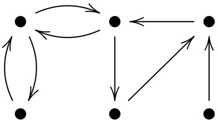

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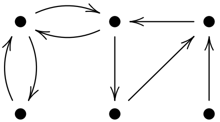

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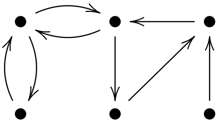

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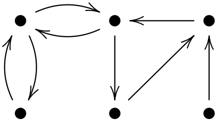

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$$P(t) = 1 - 2t^2 - t^3 + t^5$$

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Then we define the growth rate of Γ as :

$$r(\Gamma) := \limsup \sqrt[n]{C(\Gamma, n)}$$

where $C(\Gamma, n)$ is the number of closed paths of length n.

Computing entropy: the infinite clique polynomial

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Theorem

Let $\sigma \leq 1$. Then $P(t)$ defines a holomorphic function in the unit disk, and its root of minimum modulus is r^{-1}.

How to compute the core entropy without knowing complex dynamics

Let $\theta \in \mathbb{R}/\mathbb{Z}$, and $\theta_i := 2^{i-1}\theta \mod 1$, and consider the diameter $\{\theta/2, (\theta+1)/2\}$ (= major leaf).

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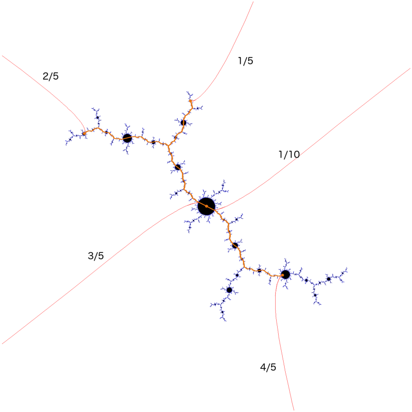

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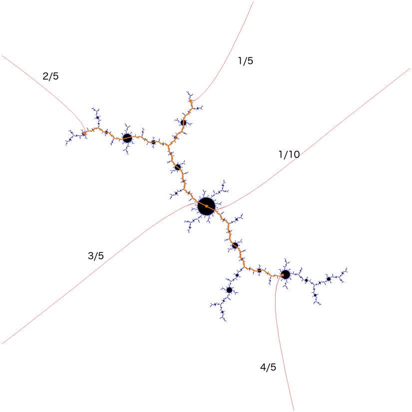

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- (i, j) non-separated $\Leftrightarrow$ θ_i and θ_j lie on same side of diam.

Wedges

$$
\begin{array}{ccccc}
 & & & & \cdots \\
 & & & (4,5) & \cdots \\
 & & (3,4) & (3,5) & \cdots \\
 & (2,3) & (2,4) & (2,5) & \cdots \\
(1,2) & (1,3) & (1,4) & (1,5) & \cdots
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Label all pairs as either separated or non-separated

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(The boxed pairs are the separated ones.)

From wedges to graphs

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$$\begin{tikzcd}
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(1,2) \arrow[ur] & \boxed{(1,3)} \arrow[l] \arrow[r] & (1,4) & \cdots
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so	$P_{\theta_n}(t) \to P_\theta(t)$	(spectral determinants)
and	$r(\theta_n) \to r(\theta)$	(growth rates)

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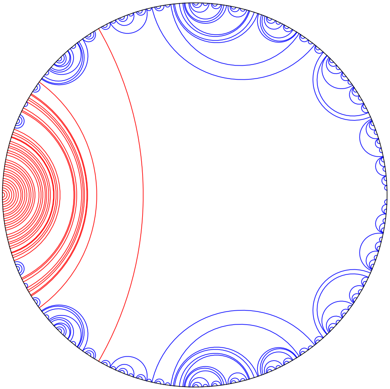
5. Higher degree polynomials

The end

Thank you!

Thurston's quadratic minor lamination (QML)



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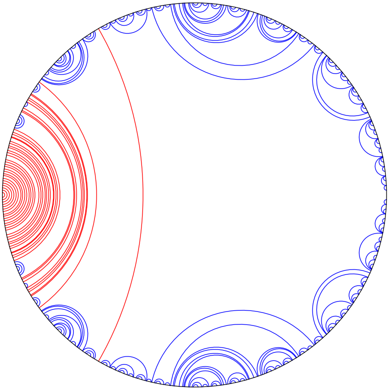

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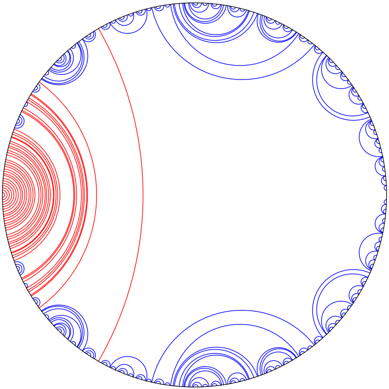
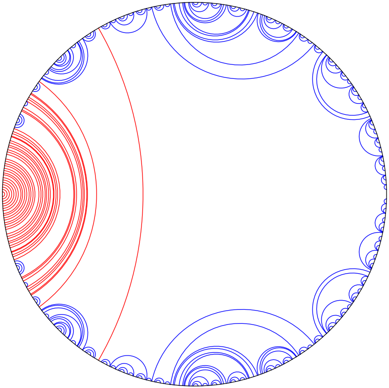


For each f_c, pick the minor leaf of the lamination for f_c (i.e, the ray pair landing at the critical value (or its root)). The QML is the union of all minor leaves for all $c \in \mathcal{M}$. The quotient $\mathcal{M}_{abs}$ of the disk by the lamination is a (locally connected) model for the Mandelbrot set, and homeomorphic to it if MLC holds.

A transverse measure on QML



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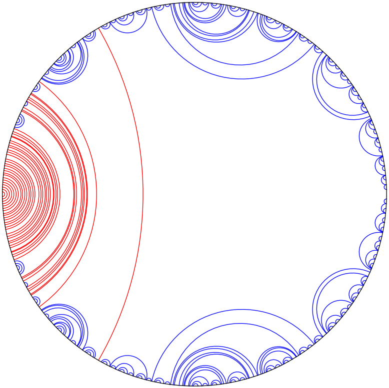

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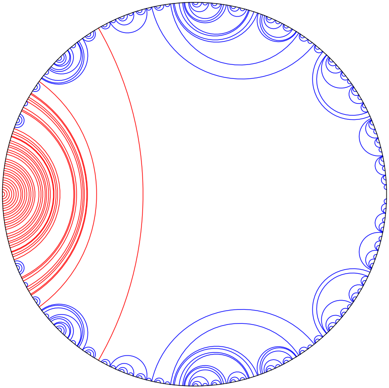

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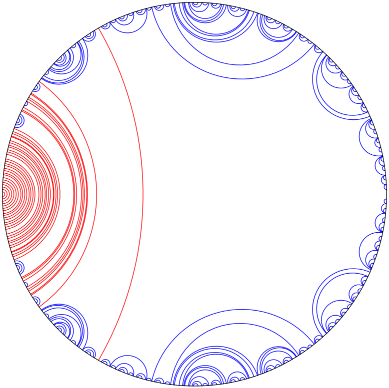

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"Combinatorial bifurcation measure"?

The end (really!)

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