1. NOTATIONS

$f_c: z \mapsto z^2 + c, z \in \mathbb{C}.$

$f^n = f$ iterated $n$ times.

$K_c = \{ z | f^n_c(z) \neq \infty \}$ the filled-in Julia set.

$J_c = \partial K_c$ the Julia set

$f_c$ has two fixed points $\alpha(c)$ and $\beta(c)$.

Convention: $\beta(c)$ is the most repulsive (the one on the right).

$M = \{ c | K_c \text{ is connected} \} = \{ c | 0 \in K_c \}$.

$\mathcal{V}_0 = \{ c | 0 \text{ is periodic for } f_c \}$ (centers of hyperbolic components).

$\mathcal{V}_2 = \{ c | 0 \text{ is strictly preperiodic} \}$ (Misiurewicz points).

$\mathcal{V}_1 = \{ c | f_c \text{ has a rational neutral cycle} \}$ (roots of hyperbolic components).
2. POTENTIAL AND EXTERNAL ARGUMENTS

The potential $G_c$ created by $K_c$ is given by

$$G_c(z) = \lim_{n \to \infty} \frac{1}{2^n} \log^+ |f_c^n(z)| = \begin{cases} 0 & \text{if } z \in K_c \\ \log |z| + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \log \left| 1 + \frac{c}{f_c^n(z)^2} \right| & \text{else} \end{cases}$$

The map $z \mapsto \phi_c(z) = \lim(f_c^n(z))^{1/2^n}$ is well defined for all $z \in \mathbb{C} - K_c$ if $K_c$ is connected, only for $G_c(z) > G_c(0)$ if $K_c$ is a Cantor set. The external argument with respect to $K_c$ is $\arg_c(z) = \arg(\phi_c(z))$. (The unit for arguments is the whole turn, not the radian.) For $z \in J_c = \partial K_c$, one can define one value of $\arg_c(z)$ for each way of access to $z$ in $\mathbb{C} - K_c$.

The external ray $R(c,\theta) = \{z|\arg_c(z) = \theta\}$ is orthogonal to the equipotential lines.

The potential created by $M$ is $G_M(c) = G_c(c)$. The conformal mapping $\phi_M: \mathbb{C} - M \to \mathbb{C} - \overline{B}$ is given by

$$\phi_M(c) = \phi_c(c)$$

This is the magic formula which allows one to get information in the parameter plane. This gives

$$\arg_M(c) = \arg_c(c) \quad \text{for } c \notin M.$$ 

The formula extends to the Misiurewicz points. For $c \in \mathcal{D}_0$, the situation is more subtle.

3. HOW TO COMPUTE $\arg_c(z)$ FOR $c \in \mathcal{D}_0 \cup \mathcal{D}_2$, z PREPERIODIC

Set $x_0 = 0$, $x_i = f_c^i(0)$, $z_1 = z$, $z_j = f_c^{j-1}(z)$, $\beta = \beta(c)$, $\beta' = -\beta(c)$. Join $\beta, \beta'$, the points $x_i$ and the points $z_j$ by arcs which remain in $K_c$. If such an arc has to cross a
component of $U$ of $K_c$, let it go straight to the center and then to the exit (the center of $U$ is the point of $U$ which is in the inverse orbit of 0, "straight" is relative to the Poincaré metric of $U$). You obtain a finite tree. Drawing this tree requires some understanding of the set $K_c$ and its dynamics, but then it is enough to compute the required angle. Choose an access $\zeta$ to $z$, and let $\zeta^j$ be the corresponding access to $z_j$. The spine of $K_c$ is the arc from $\beta$ to $\beta'$. Mark 0 each time $C$ is above the spine and 1 each time it is below. You obtain the expansion in base 2 of the external argument $\theta$ of $z$ by $\zeta$. This simply comes from the two following facts:

a) $0 < \theta < 1/2$ if $\zeta$ is above the spine, $1/2 < \theta < 1$ if it is below;

b) $f_c$ doubles the external arguments with respect to $K_c$, as well as the potential, since $\phi_c$ conjugates $f_c$ to $z \mapsto z^2$.

Note that if $c$ and $z$ are real, the tree reduces to the segment $[\beta', \beta]$ of the real line, and the sequence of 0 and 1 obtained is just the kneading sequence studied by Milnor and Thurston (except for convention: they use 1 and -1). This sequence appears now as the binary expansion of a number which has a geometrical interpretation.

4. EXTERNAL ARGUMENTS IN $M$

If $c \in \mathcal{D}_2$, then the external arguments of $c$ in $M$ are the external arguments of $c$ in $K_c$. Their number is finite, and they are rationals with even denominators.

A point $c_0 \in \mathcal{D}_0$ is in the interior of $M$, thus has no external arguments. But the corresponding point $c_1 \in \mathcal{D}_1$ (the root of the hyperbolic component whose center is $c_0$) has
2 arguments $\theta_+$ and $\theta_-$, which are rational with odd denominators. They can be obtained as follows: Let $U_0$ be the component of the interior of $K_{c_0}$ containing 0 and $U_1 = f^1_c(U_0)$, so that $U_1$ contains $c_0$. On the boundary of $U_0$, there is a periodic point $a_0$ whose period divides $k = \text{period of } 0$. Then $\theta_+$ and $\theta_-$ are the external arguments of $a_1 = f_{c_0}(a_0)$ corresponding to accesses adjacent to $U_1$.

5. USE OF EXTERNAL ARGUMENTS

We write $\theta \sim_c \theta'$ if the external rays $R(c, \theta)$ and $R(c, \theta')$ land in the same point of $K_c$, i.e., if $\theta$ and $\theta'$ are two external arguments of one point in $J_c$. For $c \in D_0 \cup D_2$, the classes having 3 elements or more is made of rational points, each class with 2 elements is limit of classes made of rational points. Knowing this equivalence relation, one can describe $K_c$ as follows: start from a closed disc, and pinch it so as to identify the points of argument $\theta$ and $\theta'$ each time you have $\theta \sim_c \theta'$. You end up with a space homeomorphic to $K_c$. A similar description can be given for $M$. It will be valid if we know that $M$ is locally connected (a fact which is highly suggested by the many pictures of details of $M$ that we have).
6. INTERNAL AND EXTERNAL ARGUMENTS IN $3\mathbb{W}_0$

$\mathbb{W}_0$ denotes the main component of $\mathbb{M}$ (the big cardioid). Internal arguments in $\mathbb{W}_0$ are defined using a conformal representation of $\mathbb{W}_0$ onto the unit disc $D$. The internal argument $\text{Arg}_{\mathbb{W}_0}(c)$ is just the argument of $f'(a,c)$ (in fact the argument of $\alpha(c)$). If a point $c \in 3\mathbb{W}_0$ has a rational internal argument $t = p_0/q_0$ (irreducible form), a compliment $\mathbb{W}_t$ of period $q_0$ is attached at $\mathbb{W}_0$ at the point $c$. Thus $c$ has 2 external arguments: $\theta_- = a_-/2^{q_0} - 1$ and $\theta_+ = a_+/2^{q_0} - 1$.

**Theorem 1.**

$$\theta_+(t) = \sum_{s<t} \frac{1}{2^{q(s)}} - 1 = \sum_{0<p/q<t} \frac{1}{2^{q(s)}}.$$  

$\theta_+(t)$ = same with $\leq t$ instead of $< t$. (Here $p(s)/q(s)$ is the irreducible representation of the rational number $s$. In the second sum, all representations are allowed.)

**Proof.** Clearly $\theta_+(t) - \theta_-(t) \geq 1/2^{q(t)} - 1$

$$\theta_-(t) \geq \sum_{s<t} \frac{1}{2^{q(s)}} - 1.$$ 

$$1 - \theta_+(t) \geq \sum_{t<s<1} \frac{1}{2^{q(s)}} - 1.$$ 

**Lemma.** $\sum_{s \in (0,1) \cap \mathbb{Q}} \frac{1}{2^{q(s)}} - 1 = 1$.

Therefore the inequalities above are equalities, which proves the theorem.

**Proof of Lemma.** Consider the integer points $(q,p)$ with $0 < p < q$, and provide each $(q,p)$ with the weight $1/2^p$. Summing on horizontal lines gives total weight = 1. Summing on
rational lines through 0 gives total weight $= \sum_{t} 1/2^{q(t)} - 1$. 

**Corollary.** The set of values of $\theta$ such that $R(M,\theta)$ lands on $\partial W_0$ has measure 0.

7. TUNING

Let $W$ be a hyperbolic component of $M$, of period $k$, and $c_0$ the center of $W$. There is a copy $M_W$ of $W$, sitting in $M$, and in which $W$ corresponds to the main cardioid $W_N$. This is particularly striking for a primitive component, and was observed by Mandelbrot in 1980. More precisely, there is a continuous injection $\psi_W : M \to M$ such that $\psi_W(0) = c_0$, $\psi_W(W) = W$, $\psi_W(M) = M_W$, $\partial M_W \subset \partial M$. For $x \in M$, the point $\psi_W(x)$ will be called "c_0 tuned by x" and denoted $c_0 \perp x$ or $W \perp x$. The filled-in Julia set $K_{c_0 \perp x}$ can be obtained in taking $K_{c_0}$ and replacing, for component $U$ of $K_{c_0}$, the part $\overline{U}$ (which is homeomorphic to the closed disc $D$) by a copy of $K_x$.

**Theorem 2.** Let $\theta_-$ and $\theta_+$ be the two external arguments in $M$ of the root $c_1$ of $W$, and let $t$ be an external argument of $x$ in $M$. Then to $t$ there corresponds an external argument $t'$ of $c_0 \perp x$ in $M$, which can be obtained by the following algorithm:

Expand $\theta_-$, $\theta_+$ and $t$ in base 2:

- $\theta_- = \underbrace{u_0 u_0 \ldots u_0}_{k} = u_1 u_2 \ldots u_k u_1 \ldots$
- $\theta_+ = \underbrace{u_1 u_1 \ldots u_1}_{k}$
- $t = \underbrace{s_1 s_2 \ldots s_n}_{\ldots}$

Then
We denote this algorithm by \( t' = (\theta_-, \theta_+) \cdot t \). According to the principle: "You plough in the \( z \)-plane and harvest in the parameter plane," this theorem relies on Proposition 1 below.

Let \( U_1 \) be the component of \( \hat{K}_{c_0} \) which contains \( x_1 = x_0 \), and \( \alpha_1 \) the root of \( U_1 \) (the point on \( \beta U_1 \) which is repulsive periodic of period dividing \( k \)). Recall that \( \theta_+ \) and \( \theta_- \) are the external arguments of \( \alpha_1 \) in \( K_{c_0} \) corresponding to the accesses adjacent to \( U_1 \).

**Proposition 1.** Let \( z \) be a point in \( \beta U_1 \) with internal argument \( t \). Then \( z \) has an external argument \( t' \) in \( M \) given by

\[
   t' = u_1^{s_1} u_2^{s_2} u_3^{s_3} \ldots
\]

**Sketch of Proof of Proposition 1.** Let \( a_1' \) be the point in \( U_1 \) opposite to \( \alpha_1 \) (point of internal argument \( 1/2 \)) and call the geodesic \( [a_1', a_1] \) the spine of \( U_1 \). Let \( U_i \) be the connected component in \( \hat{K}_{c_0} \) containing \( x_i = f_{c_0}^i(0) \), so that \( U_i = f_{c_0}^{i-1}(U_1) \), and define the spine of \( U_i \) as the image of the spine of \( U_1 \). Recall that the spine of \( K_{c_0} \) is the arc \( [\beta, \beta'] \) in \( K_{c_0} \).

**Lemma.** For each \( i \), either \( U_i \) is off the spine of \( K_{c_0} \), either the spine of \( U_i \) is the trace in \( U_i \) of the spine of \( K_{c_0} \).

We don't prove this lemma here. Now, if the first digit \( s_1 \) of \( t \) is 0, \( z \) is on the same side of the spine of \( U_1 \) as the ray \( R(\theta_-) \). Then \( z_i = f_{c_0}^{i-1}(z) \) will be on the same side of the spine of \( U_i \) as \( R(z_1^{-1} \theta_-) \) (thus also on the same side of the spine of \( K_{c_0} \)) for \( i = 1, \ldots, k \). Therefore the \( i \)-th digit \( s_i' \) of \( t' \) is \( u_1^{s_1} \). If \( s_1 = 1 \), then \( z \) follows \( \theta_+ \), and \( s_i' = u_i^{1} \) for
i = 1, ..., k. At time k+1, \(z_{k+1}\) is back in \(U_1\), but with internal argument 2t, so that \(s_1\) is replaced by \(s_2\) (the internal argument is preserved in the map from \(U_i\) to \(U_{i+1}\) for \(i = 1, ..., k-1\), and doubled from \(U_k = U_0\) to \(U_1\)). And we start for a second run, and so on...

Now let us come to the situation of Theorem 2. There is a homeomorphism \(\psi\) of \(K_x\) onto the part of \(K_{c_0 \perp x}\) which corresponds to \(U_1\) in \(K_{c_0}\). This homeomorphism conjugates \(f_x\) to \(f_{c_0 \perp x}^k\).

**Lemma 2.** If \(y \in J_x = \partial K_x\) has external argument \(t\), then the external arguments of \(\psi(y)\) in \(K_{c_0 \perp x}\) are the same as the external arguments in \(K_{c_0}\) of the point \(y' \in \partial U_1\) whose internal argument is \(t\).

We don't prove this lemma here, but if you think of it, it is very natural. Theorem 2 now follows from results discussed in Section 4, applied to \(y = x\) or to \(y = \) the root of the component \(V_1\) of \(K_x\) containing \(x\).

**Remark.** The copy \(M_w\) of \(M\) sits in \(M\), but in some places, where \(M_w\) ends \(M\) goes on. Then there are points \(x\) in \(M\) with 1 external argument such that \(c_0 \perp x\) has several external arguments. How is this compatible with the algorithm described in Theorem 2? Well, this algorithm is not univalent: It starts by expanding \(t\) in base 2, and numbers of the form \(p/2^k\) have two expansions. Actually \(c_0 \perp x\) may have more than 2 external arguments; the algorithm will give the two which correspond to accesses adjacent to \(M_w\).
Example

\[ 1/2^n = .00...01 \]

An elephant in \( M \).

Its copy in \( W \) has two smokes coming out of its trunk if \( W \) is the rabbit component.

8. FEIGENBAUM POINT AND MORSE NUMBER

Let \( W_n \) be the \( n \)-th component of \( \hat{M} \) in the Myrberg-Feigenbaum cascade: \( W_1 \) is the disc \( D(-1, 1/4) \) and we have \( W_n = W_1 \perp W_1 \perp \ldots \perp W_1 \). Denote \( c_0(n) \) and \( c_1(n) \) the center and the root of \( W_n \). According to Feigenbaum theory, \( c_0(n) \) converges to \( c_\infty = -1.401\ldots \) exponentially with ratio \( 1/4.66\ldots \). The external arguments \( \theta_-(n) \) and \( \theta_+(n) \) of \( c_1(n) \) are obtained by Theorem 2, starting from

\[ \theta_- (1) = 1/3 = .01 \quad \text{and} \]

\[ \theta_+ (1) = 2/3 = .10 . \]
One gets:
\[ \theta_-(2) = .\overline{0110} \]
\[ \theta_+(2) = .\overline{1001} \]
\[ \theta_-(3) = .\overline{01101001} \]
\[ \theta_+(3) = .\overline{10010110} \]
\[ \theta_-(4) = .\overline{0110100110010110} \]

... The numbers \( \theta_-(n) \) converge to a number \( \theta_-(\infty) \) known as the Morse number. (It has been proved to be transcendental by Van der Poorten, an Australian number theorist in Bordeaux.

The idea is that a sequence of digits which represents an element in \( \mathbb{Z}/2 \mathbb{[T]} \) which is algebraic over \( \mathbb{Z}/2 \mathbb{T} \) cannot be the expansion in base 2 of an algebraic real number, except if periodic.) Note that the convergence of \( \theta_-(n) \) to \( \theta_-(\infty) \) is faster than any exponential convergence: the number of good digits is doubled at each time. In our view, this is due to the fact that, because of the growing of hairs, \( W_{n+1} \) is more sheltered from Brownian dust by \( W_n \) and \( W_n \) is by \( W_{n-1} \).

9. SPIRALING ANGLE

Take \( c \in \mathbb{C} \) (in \( M \) or in \( \mathbb{C} - M \)), and let \( z_0 \) be a repulsive periodic point for \( f_c \), of period \( k \) and multiplier \( \rho = (f_c^k)'(z_0) \).

There are external rays of \( K_c \) landing on \( z_0 \) (there may be a finite number or an infinity of them, almost always a finite number though). Take one of them \( R \) and let \( \tilde{R} \) be an image of \( R \) by a determination of \( z + \log(z-z_0) \). Recall that, with our
convention, \( \log z = \log|z| + 2\pi i \arg z \). When \( z \to 0 \) on \( \mathbb{R} \),
\( w = \log z \to \infty \) in \( \mathbb{C} \) and \( \text{Im } w/\text{Re } w = 2\pi \arg|z - z_0|/\log|z - z_0| \) has
a limit \( m \) that we call the **spiraling slope** of \( z_0 \). This slope
can be written \( m = \frac{2\pi \sigma}{\log|\rho|} \), and \( \sigma \) will be the **spiraling number**
(or spiraling angle). We have \( \sigma = \arg \rho - \omega \), where \( \omega \) is the
rotation angle of the action of \( f^k \) on the set of external
rays landing at \( z_0 \). If there are \( v \) such rays, then \( \omega \) is of
the form \( p/v \) with \( p \in \mathbb{Z} \). Note that \( \arg \rho \) and \( \omega \) are but
angles, i.e., \( \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \), while \( \sigma \) is naturally in \( \mathbb{R} \).

In order to compute \( \omega \) and \( \sigma \), let \( c \) vary in a simply con­
nected domain \( \Lambda \subset \mathbb{C} \) and \( z_0(c) \) vary accordingly, remaining
repulsive periodic of period \( k \). We make the following observa­
tions:

1. \( \omega \) is continuous in \( c \), and is invariant under the Hubbard­
Branner stretching procedure (here it means just sliding \( c \)
along the external rays of \( M \)).
2. \( \omega \) remains constant in \( \Lambda \) if the number of external rays
landing in \( z_0(c) \) is finite and constant.
3. \( \sigma \) tends to 0 when \( |p| \to 1 \), i.e., when \( z_0 \) turns indif­
ferent periodic (note that \( m \) does not necessarily tend to
0, and may well tend to \( \infty \) so that you see the Julia set
spiraling a lot).

If we take \( c \in \mathbb{C} - W_0 \) and \( z_0(c) = \alpha(c) \), the least repulsive
fixed point of \( f_c \), we obtain the following: is the internal
argument in \( W_0 \) of \( \pi_{W_0}(c) \), which can be defined in the following
way if you admit that \( M \) is locally connected. If \( c \in M \),
\( \pi_{W_0}(c) \) is the point where an arc in \( M \) from \( c \) to 0 enters \( W_0 \).
If \( c \notin M \), let \( \pi_M(c) \) be the point where the external ray of \( M \)
through \( c \) lands in \( M \). Then \( \pi_{W_0}(c) = \pi_{W_0}(\pi_M(c)) \). In fact,
one can modify this definition so that it does not depend on
the local connectivity of $M$, and determines $\pi \omega_0(c)$ unambig-
ously.

10. HOW TO DETERMINE $\pi \omega_0(c)$ KNOWING 1 PAIR $(t,t')$ SUCH THAT
$t \sim_c t'$, $t \neq t'$

Here is the algorithm: expand $t$ and $t'$ in base 2:

\[ t = \ldots u_1 u_2 \]

\[ t' = \ldots u'_1 u'_2 \]

Set $\delta_i = u_i - u'_i \mod 2$. If the sequence $\delta_i$ ends in 1,0,0,0,...,
i.e., $\delta_i = 0$ for $i \geq n$, $\delta_{n-i} = 1$, then $\theta = 2^n t = 2^n t'$ is the
external argument of $c$ in $M$, and the internal argument of
$\pi \omega_0(c)$ is the angle $t \pmod{0}$ such that $0 < \theta < 0 + (t \pmod{0})$ (the
functions $\theta$ and $\theta$ are defined in Section 6). If the sequence
$\delta_i$ ends with 1111..., then $\pi \omega_0(c) = -3/4$ (internal argument
1/2). Else, look for 10 somewhere in the sequence, i.e.,
$\delta_{n-1} = 1$, $\delta_n = 0$. Then there is a $t_0$ such that $\theta = 2^n t$ and
$\theta' = 2^n t'$ both belong to $[\theta_-(t_0), \theta_+(t_0)]$, and that is the
internal argument of $\pi \omega_0(c)$. Why? Let $z$ be the point in $K_c$
with $t$ and $t'$ as external arguments. The rays $R(2^{n-1} t)$ and
$R(2^{n-1} t')$ are each on one side of the spine of $K_c$, therefore
$f_c^{n-1}(z)$ belongs to this spine. Now the image of the spine
$[\beta, \beta']_{K_c}$ is the arc $[\beta, c]_{K_c}$ from $\beta$ to $c$ in $K_c$, and because
$R(2^n t)$ and $R(2^n t')$ are on the same side of the spine, neces-
sarily $f_c^n(z) \in [\alpha(c), c]_{K_c}$. From this observation the result
follows easily.
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