ALGORITHMS FOR COMPUTING ANGLES IN THE MANDELBROT SET

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1. NOTATIONS

$f_c: z \mapsto z^2 + c$, $z \in \mathbb{C}$.

f^n = f iterated n times.

$K_c = \{z \mid f_c^n(z) \not\to \infty\}$ the filled-in Julia set.

$J_c = \partial K_c$ the Julia set

f_c has two fixed points $\alpha(c)$ and $\beta(c)$.

Convention: $\beta(c)$ is the most repulsive (the one on the right).

$M = \{c \mid K_c$ is connected$\} = \{c \mid 0 \in K_c\}$.

$\mathcal{D}_0 = \{c \mid 0$ is periodic for $f_c\}$ (centers of hyperbolic components).

$\mathcal{D}_2 = \{c \mid 0$ is strictly preperiodic$\}$ (Misiurewicz points).

$\mathcal{D}_1 = \{c \mid f_c$ has a rational neutral cycle$\}$ (roots of hyperbolic components).

Chaotic Dynamics
and Fractals

2. POTENTIAL AND EXTERNAL ARGUMENTS

The potential G_c created by K_c is given by

$$G_c(z) = \lim \frac{1}{2^n} \operatorname{Log}^+|f_c^n(z)| = \begin{cases} 0 & \text{if } z \in K_c \\ \operatorname{Log}|z| + \Sigma \frac{1}{2^{n+1}} \operatorname{Log}\left|1 + \frac{c}{f^n(z)^2}\right| & \text{else} \end{cases}$$

The map $z \mapsto \phi_c(z) = \lim(f_c^n(z))^{1/2^n}$ is well defined for all $z \in \mathbb{C} - K_c$ if K_c is connected, only for $G_c(z) > G_c(0)$ if K_c is a Cantor set. The external argument with respect to K_c is $\operatorname{Arg}_c(z) = \operatorname{Arg}(\phi_c(z))$. (The unit for arguments is the whole turn, not the radian.) For $z \in J_c = \partial K_c$, one can define one value of $\operatorname{Arg}_c(z)$ for each way of access to z in $\mathbb{C} - K_c$. The external ray $R(c,\theta) = \{z|\operatorname{Arg}_c(z) = \theta\}$ is orthogonal to the equipotential lines.

The potential created by M is $G_M(c) = G_c(c)$. The conformal mapping ϕ_M: $\mathbb{C} - M \to \mathbb{C} - \bar{D}$ is given by

$$\phi_M(c) = \phi_c(c)$$

This is the magic formula which allows one to get information in the parameter plane. This gives

$$\operatorname{Arg}_M(c) = \operatorname{Arg}_c(c) \quad \text{for } c \notin M.$$

The formula extends to the Misiurewicz points. For $c \in \mathcal{D}_0$, the situation is more subtle.

3. HOW TO COMPUTE $\operatorname{Arg}_c(z)$ FOR $c \in \mathcal{D}_0 \cup \mathcal{D}_2$, z PREPERIODIC

Set $x_0 = 0$, $x_i = f_c^i(0)$, $z_1 = z$, $z_j = f^{j-1}(z)$, $\beta = \beta(c)$, $\beta' = -\beta(c)$. Join β, β', the points x_i and the points z_j by arcs which remain in K_c. If such an arc has to cross a component of U of $\mathring{K}_c$, let it go straight to the center and then to the exit (the center of U is the point of U which is in the inverse orbit of 0, "straight" is relative to the Poincaré metric of U). You obtain a finite tree. Drawing this tree requires some understanding of the set K_c and its dynamics, but then it is enough to compute the required angle. Choose an access ζ to z, and let ζ^j be the corresponding access to z_j. The spine of K_c is the arc from β to β'. Mark 0 each time ζ^j is above the spine and 1 each time it is below. You obtain the expansion in base 2 of the external argument θ of z by ζ. This simply comes from the two following facts:

a) $0 < \theta < 1/2$ if ζ is above the spine, $1/2 < \theta < 1$ if it is below;

b) f_c doubles the external arguments with respect to K_c, as well as the potential, since ϕ_c conjugates f_c to $z \mapsto z^2$.

Note that if c and z are real, the tree reduces to the segment $[\beta',\beta]$ of the real line, and the sequence of 0 and 1 obtained is just the kneading sequence studied by Milnor and Thurston (except for convention: they use 1 and -1). This sequence appears now as the binary expansion of a number which has a geometrical interpretation.

4. EXTERNAL ARGUMENTS IN M

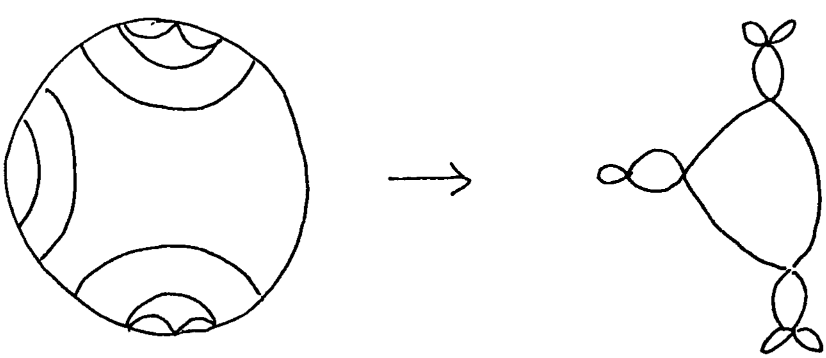
If $c \in \mathcal{D}_2$, then the external arguments of c in M are the external arguments of c in K_c. Their number is finite, and they are rationals with even denominators.

A point $c_0 \in \mathcal{D}_0$ is in the interior of M, thus has no external arguments. But the corresponding point $c_1 \in \mathcal{D}_1$ (the root of the hyperbolic component whose center is c_0) has

2 arguments θ_+ and θ_-, which are rational with odd denominators. They can be obtained as follows: Let U_0 be the component of the interior of K_{c_0} containing 0 and $U_i = f_c^i(U_0)$, so that U_1 contains c_0. On the boundary of U_0, there is a periodic point α_0 whose period divides k = period of 0. Then θ_+ and θ_- are the external arguments of $\alpha_1 = f_{c_0}(\alpha_0)$ corresponding to accesses adjacent to U_1.

5. USE OF EXTERNAL ARGUMENTS

We write $\theta \sim_c \theta'$ if the external rays $R(c,\theta)$ and $R(c,\theta')$ land in the same point of K_c, i.e., if θ and θ' are two external arguments of one point in J_c. For $c \in \mathcal{D}_0 \cup \mathcal{D}_2$, the classes having 3 elements or more is made of rational points, each class with 2 elements is limit of classes made of rational points. Knowing this equivalence relation, one can describe K_c as follows: start from a closed disc, and pinch it so as to identify the points of argument θ and θ' each time you have $\theta \sim_c \theta'$. You end up with a space homeomorphic to K_c. A similar description can be given for M. It will be valid if we know that M is locally connected (a fact which is highly suggested by the many pictures of details of M that we have).



6. INTERNAL AND EXTERNAL ARGUMENTS IN ∂W_0

W_0 denotes the main component of M (the big cardioid). Internal arguments in W_0 are defined using a conformal representation of W_0 onto the unit disc D. The internal argument $\mathrm{Arg}_{W_0}(c)$ is just the argument of $f_c'(\alpha,c))$ (in fact the argument of $\alpha(c)$). If a point $c \in \partial W_0$ has a rational internal argument $t = p_0/q_0$ (irreducible form), a complment W_t of period q_0 is attached at W_0 at the point c. Thus c has 2 external arguments

$$\theta_- = a_-/2^{q_0} - 1 \quad \text{and} \quad \theta_+ = a_+/2^{q_0} - 1.$$

Theorem 1.

$$\theta_-(t) = \sum_{s<t} 1/2^{q(s)} - 1 = \sum_{0<p/q<t} 1/2^q .$$

$\theta_+(t)$ = same with $\le t$ instead of $< t$. (Here $p(s)/q(s)$ is the irreducible representation of the rational number s. In the second sum, all representations are allowed.)

Proof. Clearly $\theta_+(t) - \theta_-(t) \ge 1/2^{q(t)} - 1$

$$\theta_-(t) \ge \sum_{s<t} 1/2^{q(s)} - 1.$$

$$1 - \theta_+(t) \ge \sum_{t<s<1} 1/2^{q(s)} - 1.$$

Lemma. $\displaystyle\sum_{s\in(0,1)\cap\mathbb{Q}} 1/2^{q(s)} - 1 = 1.$

Therefore the inequalities above are equalities, which proves the theorem.

Proof of Lemma. Consider the integer points (q,p) with $0 < p < q$, and provide each (q,p) with the weight $1/2^p$. Summing on horizontal lines gives total weight = 1. Summing on rational lines through 0 gives total weight $= \sum_t 1/2^{q(t)} - 1$.

<u>Corollary</u>. The set of values of θ such that $R(M,\theta)$ lands on ∂W_0 has measure 0.

7. TUNING

Let W be a hyperbolic component of M, of period k, and c_0 the center of W. There is a copy M_W of , sitting in M, and in which W corresponds to the main cardioid W_0. This is particularly striking for a primitive component, and was observed by Mandelbrot in $\sim$1980. More precisely, there is a continuous injection $\psi_W : M \to M$ such that $\psi_W(0) = c_0$, $\psi_W(W_0) = W$, $M_W = \psi_W(M)$, $\partial M_W \subset \partial M$. For $x \in M$, the point $\psi_W(x)$ will be called "c_0 tuned by x" and denoted $c_0 \perp x$ or $W \perp x$. The filled-in Julia set $K_{c_0 \perp x}$ can be obtained in taking K_{c_0} and replacing, for component U of $\mathring{K}_{c_0}$, the part $\bar{U}$ (which is homeomorphic to the closed disc $\bar{D}$) by a copy of K_x.

<u>Theorem 2</u>. Let θ_- and θ_+ be the two external arguments in M of the root c_1 of W, and let t be an external argument of x in M. Then to t there corresponds an external argument t' of $c_0 \perp x$ in M, which can be obtained by the following algorithm:

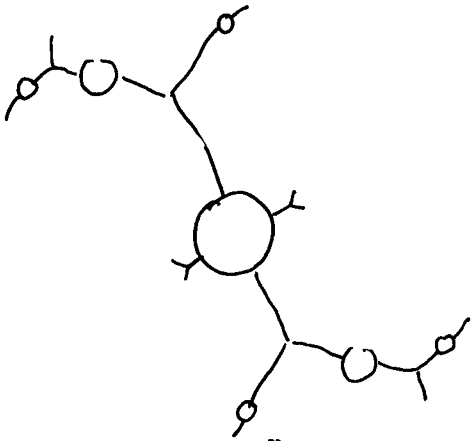
Expand θ_-, θ_+ and t in base 2:

$$\theta_- = .\overline{u_1^0 u_2^0 \ldots u_k^0} = .u_1^0 u_2^0 \ldots u_k^0 u_1^0 \ldots u_k^0 u_1^0 \ldots$$

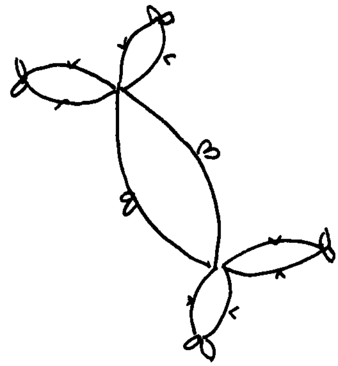
$$\theta_+ = .\overline{u_1^1 u_2^1 \ldots u_k^1}$$

$$t = .s_1 s_2 \ldots s_n \ldots .$$

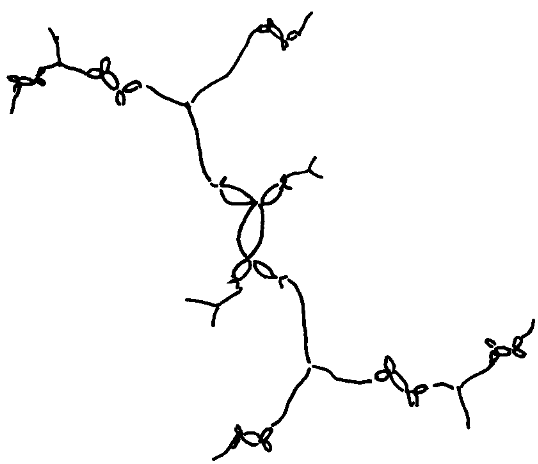
Then



K_{c_0}



K_x



$K_{c_0} \perp x$

$$t' = .u_1^{s_1} \dots u_k^{s_1} u_1^{s_2} \dots u_k^{s_2} u_1^{s_3} \dots$$

We denote this algorithm by $t' = (\theta_-, \theta_+) \perp t$. According to the principle: "You plough in the z-plane and harvest in the parameter plane," this theorem relies on Proposition 1 below. Let U_1 be the component of $\mathring{K}_{c_0}$ which contains $x_1 = x_0$, and α_1 the root of U_1 (the point on ∂U_1 which is repulsive periodic of period dividing k). Recall that θ_+ and θ_- are the external arguments of α_1 in K_{c_0} corresponding to the accesses adjacent to U_1.

Proposition 1. Let z be a point in ∂U_1 with internal argument t. Then z has an external argument t' in M given by $t' = (\theta_-, \theta_+) \perp t$.

Sketch of Proof of Proposition 1. Let α_1' be the point in U_1 opposite to α_1 (point of internal argument 1/2) and call the geodesic $[\alpha_1, \alpha_1']$ the spine of U_1. Let U_i be the connected component in $\mathring{K}_{c_0}$ containing $x_i = f_{c_0}^i(0)$, so that $U_i = f_{c_0}^{i-1}(U_1)$, and define the spine of U_i as the image of the spine of U_1. Recall that the spine of K_{c_0} is the arc $[\beta, \beta']$ in K_{c_0}.

Lemma. For each i, either U_i is off the spine of K_{c_0}, either the spine of U_i is the trace in $\bar{U}_i$ of the spine of K_{c_0}.

We don't prove this lemma here. Now, if the first digit s_1 of t is 0, z is on the same side of the spine of U_1 as the ray $R(\theta_-)$. Then $z_i = f_{c_0}^{i-1}(z)$ will be on the same side of the spine of U_i as $R(2^{i-1}\theta_-)$ (thus also on the same side of the spine of K_{c_0}) for $i = 1,\dots,k$. Therefore the ith digit s_i' of t' is u_i^0. If $s_1 = 1$, then z follows θ_+, and $s_i' = u_i^1$ for

$i = 1,\dots,k$. At time k+1, z_{k+1} is back in U_1, but with internal argument 2t, so that s_1 is replaced by s_2 (the internal argument is preserved in the map from U_i to U_{i+1} for $i = 1,\dots,k-1$, and doubled from $U_k = U_0$ to U_1). And we start for a second run, and so on...

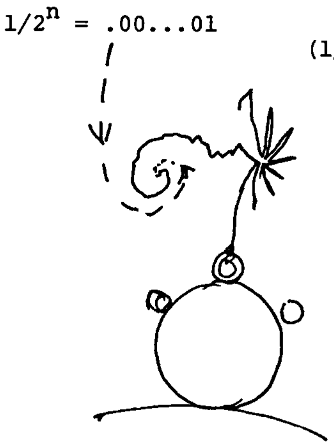
Now let us come to the situation of Theorem 2. There is a homeomorphism ψ of K_x onto the part of $K_{c_0 \perp x}$ which corresponds to $\bar{U}_1$ in K_{c_0}. This homeomorphism conjugates f_x to $f^k_{c_0 \perp x}$.

<u>Lemma 2</u>. If $y \in J_x = \partial K_x$ has external argument t, then the external arguments of $\psi(y)$ in $K_{c_0 \perp x}$ are the same as the external arguments in K_{c_0} of the point $y' \in \partial U_1$ whose internal argument is t.

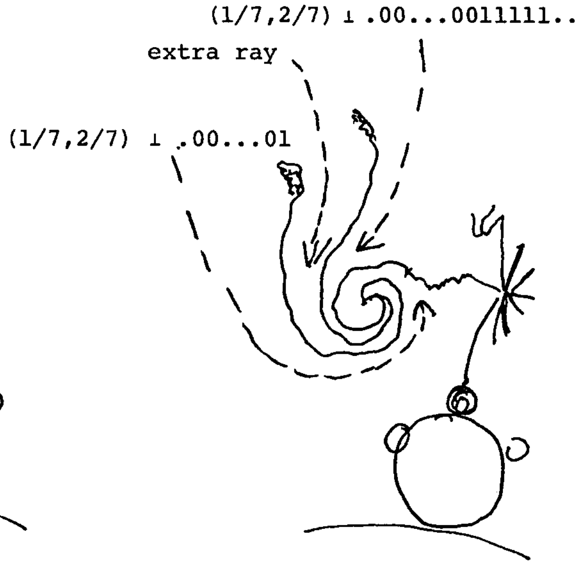
We don't prove this lemma here, but if you think of it, it is very natural. Theorem 2 now follows from results discussed in Section 4, applied to y = x or to y = the root of the component V_1 of $\mathring{K}_x$ containing x.

<u>Remark</u>. The copy M_W of M sits in M, but in some places, where M_W ends M goes on. Then there are points x in M with 1 external argument such that $c_0 \perp x$ has several external arguments. How is this compatible with the algorithm described in Theorem 2? Well, this algorithm is not univalent: It starts by expanding t in base 2, and numbers of the form $p/2^\ell$ have two expansions. Actually $c_0 \perp x$ may have more than 2 external arguments; the algorithm will give the two which correspond to accesses adjacent to M_W.

Example



An elephant in M.



Its copy in W has two smokes coming out of its trunk if W is the rabbit component.

8. FEIGENBAUM POINT AND MORSE NUMBER

Let W_n be the n-th component of $\overset{\circ}{M}$ in the Myrberg-Feigenbaum cascade: W_1 is the disc D(-1, 1/4) and we have $W_n = W_1 \perp W_1 \perp \ldots \perp W_1$. Denote $c_0(n)$ and $c_1(n)$ the center and the root of W_n. According to Feigenbaum theory, $c_0(n)$ converges to $c_\infty = -1.401\ldots$ exponentially with ratio 1/4.66... . The external arguments $\theta_-(n)$ and $\theta_+(n)$ of $c_1(n)$ are obtained by Theorem 2, starting from

$$\theta_-(1) = 1/3 = \overline{.01} \quad \text{and}$$

$$\theta_+(1) = 2/3 = \overline{.10}\ .$$

One gets:

$\theta_-(2) = .\overline{0110}$

$\theta_+(2) = .\overline{1001}$

$\theta_-(3) = .\overline{01101001}$

$\theta_+(3) = .\overline{10010110}$

$\theta_-(4) = .\overline{0110100110010110}$

....

The numbers $\theta_-(n)$ converge to a number $\theta_-(\infty)$ known as the Morse number. (It has been proved to be transcendental by Van der Pooten, an Australian number theorist in Bordeaux. The idea is that a sequence of digits which represents an element in $\mathbb{Z}/2\ [[T]]$ which is algebraic over $\mathbb{Z}/2\ (T)$ cannot be the expansion in base 2 of an algebraic real number, except if periodic.) Note that the convergence of $\theta_-(n)$ to $\theta_-(\infty)$ is faster than any exponential convergence: the number of good digits is doubled at each time. In our view, this is due to the fact that, because of the growing of hairs, W_{n+1} is more sheltered from Brownian dust by W_n and W_n is by W_{n-1}.

9. SPIRALING ANGLE

Take $c \in \mathbb{C}$ (in M or in $\mathbb{C}$ - M), and let z_0 be a repulsive periodic point for f_c, of period k and multiplier $\rho = (f_c^k)'(z_0)$. There are external rays of K_c landing on z_0 (there may be a finite number or an infinity of them, almost always a finite number though). Take one of them R and let $\tilde{R}$ be an image of R by a determination of $z \to \mathrm{Log}(z - z_0)$. Recall that, with our convention, $\operatorname{Log} z = \operatorname{Log}|z| + 2\pi i \operatorname{Arg} z$. When $z \to 0$ on R, $w = \operatorname{Log} z \to \infty$ in $\tilde{R}$ and $\operatorname{Im} w/\operatorname{Re} w = 2\pi\operatorname{Arg}|z - z_0|/\operatorname{Log}|z - z_0|$ has a limit m that we call the spiraling slope of z_0. This slope can be written $m = \frac{2\pi\sigma}{\operatorname{Log}|\rho|}$, and σ will be the spiraling number (or spiraling angle). We have $\sigma = \operatorname{Arg}\rho - \omega$, where ω is the rotation angle of the action of f^k on the set of external rays landing at z_0. If there are ν such rays, then ω is of the form p/ν with $p \in \mathbb{Z}$. Note that $\operatorname{Arg}\rho$ and ω are but angles, i.e., $\in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, while σ is naturally in $\mathbb{R}$.

In order to compute ω and σ, let c vary in a simply connected domain $\Lambda \subset \mathbb{C}$ and $z_0(c)$ vary accordingly, remaining repulsive periodic of period k. We make the following observations:

1. ω is continuous in c, and is invariant under the Hubbard-Branner stretching procedure (here it means just sliding c along the external rays of M).
2. ω remains constant in Λ if the number of external rays landing in $z_0(c)$ is finite and constant.
3. σ tends to 0 when $|p| \to 1$, i.e., when z_0 turns indifferent periodic (note that m does not necessarily tend to 0, and may well tend to ∞ so that you see the Julia set spiraling a lot).

If we take $c \in \mathbb{C} - W_0$ and $z_0(c) = \alpha(c)$, the least repulsive fixed point of f_c, we obtain the following: is the internal argument in W_0 of $\pi_{W_0}(c)$, which can be defined in the following way if you admit that M is locally connected. If $c \in M$, $\pi_{W_0}(c)$ is the point where an arc in M from c to 0 enters W_0. If $c \notin M$, let $\pi_M(c)$ be the point where the external ray of M through c lands in M. Then $\pi_{W_0}(c) = \pi_{W_0}(\pi_M(c))$. In fact, one can modify this definition so that it does not depend on the local connectivity of M, and determines $\pi_{W_0}(c)$ unambiguously.

10. HOW TO DETERMINE $\pi_{W_0}(c)$ KNOWING 1 PAIR (t,t') SUCH THAT $t \sim_c t'$, $t \neq t'$

Here is the algorithm: expand t and t' in base 2:

$$t = .u_1u_2 \ldots$$

$$t' = .u_1'u_2' \ldots$$

Set $\delta_i = u_i - u_i' \bmod 2$. If the sequence δ_i ends in $1,0,0,0,\ldots$, i.e., $\delta_i = 0$ for $i \geq n$, $\delta_{n-i} = 1$, then $\theta = 2^n\, t = 2^n\, t'$ is the external argument of c in M, and the internal argument of $\pi_{W_0}(c)$ is the angle t_0 such that $\theta_-(t_0) \leq \theta \leq \theta_+(t_0)$ (the functions θ_- and θ_+ are defined in Section 6). If the sequence δ_i ends with 1111..., then $\pi_{W_0}(c) = -3/4$ (internal argument 1/2). Else, look for 10 somewhere in the sequence, i.e., $\delta_{n-1} = 1$, $\delta_n = 0$. Then there is a t_0 such that $\theta = 2^n t$ and $\theta' = 2^n t'$ both belong to $[\theta_-(t_0), \theta_+(t_0)]$, and that is the internal argument of $\pi_{W_0}(c)$. Why? Let z be the point in K_c with t and t' as external arguments. The rays $R(2^{n-1}t)$ and $R(2^{n-1}t')$ are each on one side of the spine of K_c, therefore $f_c^{n-1}(z)$ belongs to this spine. Now the image of the spine $[\beta,\beta']_{K_c}$ is the arc $[\beta,c]_{K_c}$ from β to c in K_c, and because $R(2^n t)$ and $R(2^n t')$ are on the same side of the spine, necessarily $f_c^n(z) \in [\alpha(c),c]_{K_c}$. From this observation the result follows easily.

11. ACKNOWLEDGEMENTS

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