Twist deformation and parabolic implosion surgery for rational maps

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Contents

Semi-rational maps

$F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a branched covering with $\deg F \geq 2$.
Ω_F: the set of critical points of F.
The **post-critical set** of F is:

$$\mathcal{P}_F = \overline{\bigcup_{n \geq 0} F^n(\Omega_F)}.$$

The map F is **post-critically finite** if $\mathcal{P}_F$ is a finite set, or **geometrically finite** if the accumulation point set $\mathcal{P}'_F$ is a finite set.

A geometrically finite branched covering F is a **semi-rational map** if
(1) F is holomorphic in a neighborhood of $\mathcal{P}'_F$,
(2) each cycle in $\mathcal{P}'_F$ is either attracting or parabolic, and
(3) each attracting petal at a parabolic cycle in $\mathcal{P}'_F$ contains post-critical points.

The c-equivalence

Let F be a semi-rational map. An open set $\mathcal{U} \subset \widehat{\mathbb{C}}$ is a **fundamental set** of F if $\mathcal{U}$ contains every attracting cycles in $\mathcal{P}'_F$ and an attracting flower at every parabolic cycle in $\mathcal{P}'_F$.

Two semi-rational maps F and G are **c-equivalent** if $\exists\ \phi, \psi \in \mathrm{Hom}^+(\widehat{\mathbb{C}})$ and a fundamental set $\mathcal{U}$ of F such that:
(a) ψ is homotopic to ϕ rel $\mathcal{P}_F$,
(b) $\phi \circ F = G \circ \psi$,
(c) ϕ is holomorphic in U and $\psi = \phi$ in $\mathcal{U}$.

The Thurston type theorem

Theorem [CT1]

Let G be a semi-rational map with $\mathcal{P}'_G \neq \emptyset$. Then G is c-equivalent to a rational map g iff G has no Thurston obstruction nor connecting arc. Moreover, the rational map g is unique up to holomorphic conjugation.

[DH,1993] for $\#\mathcal{P}_F < \infty$,
[JZ, 2010] and [CT, 2011] for $\#\mathcal{P}'_F < \infty$,
[CT1], http://arxiv.org/abs/1501.01385.

Let β be a non-trivial open arc in $\widehat{\mathbb{C}} \setminus \mathcal{P}_G$ connecting parabolic periodic points z_0 and z_1 in $\mathcal{P}'_G$. β is a **connecting arc** if β is disjoint from a fundamental set of G and homotopic to a component of $G^{-p}(\beta)$ rel $\mathcal{P}_G$ for some integer $p > 0$.

Multicurves

A Jordan curve γ in $\widehat{\mathbb{C}} \smallsetminus \mathcal{P}_F$ is **essential** if both of the two components of $\widehat{\mathbb{C}} \smallsetminus \gamma$ contains at least two points of $\mathcal{P}_F$.

A **multicurve** Γ is a non-empty and finite collection of disjoint Jordan curves in $\widehat{\mathbb{C}} \smallsetminus \mathcal{P}_F$, each essential and no two homotopic rel $\mathcal{P}_F$.

A multicurve Γ is **stable** if each essential curve in $F^{-1}(\gamma)$ for $\gamma \in \Gamma$ is homotopic rel $\mathcal{P}_F$ to a curve in Γ.

Thurston obstructions

Let Γ be a multicurve of F. Its **transition matrix** $M_\Gamma = (a_{\gamma\beta})$ is defined by:

$$a_{\gamma\beta} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \beta)},$$

where the summation is taken over all essential components α of $F^{-1}(\beta)$ which are homotopic to γ rel $\mathcal{P}_F$.

A stable multicurve Γ is a **Thurston obstruction** of F if the leading eigenvalue $\lambda(\Gamma)$ of M_Γ satisfies $\lambda(\Gamma) \geq 1$.

Thurston algorithm

Let F be a semi-rational map.
θ_n: normalized homeomorphisms.
f_n: rational maps.

$$\begin{array}{ccccccccccc} \cdots \longrightarrow \widehat{\mathbb{C}} & \xrightarrow{F} & \widehat{\mathbb{C}} & \longrightarrow \cdots \longrightarrow & \widehat{\mathbb{C}} & \xrightarrow{F} & \widehat{\mathbb{C}} & \xrightarrow{F} & \widehat{\mathbb{C}} \\ \theta_n \downarrow & & \theta_{n-1} \downarrow & & \theta_2 \downarrow & & \theta_1 \downarrow & & \downarrow \mathrm{id} \\ \cdots \longrightarrow \widehat{\mathbb{C}} & \xrightarrow{f_n} & \widehat{\mathbb{C}} & \longrightarrow \cdots \longrightarrow & \widehat{\mathbb{C}} & \xrightarrow{f_2} & \widehat{\mathbb{C}} & \xrightarrow{f_1} & \widehat{\mathbb{C}} \end{array}$$

$\{f_n\}$ is called a **Thurston sequence** of F.

Theorem [CT1]

Suppose that F is c-equivalent to a rational map. Then the sequence $\{f_n\}$ converges uniformly to a rational map f which is c-equivalent to F as $n \to \infty$.

Distortion

Let $V \subset \hat{\mathbb{C}}$ be an open set and let $\phi : V \to \hat{\mathbb{C}}$ be a univalent map. Define

$$\mathscr{D}(\phi, V) = \sup_{E_1, E_2 \subset V} |\operatorname{mod} A(E_1, E_2) - \operatorname{mod} A(\phi(E_1), \phi(E_2))|,$$

where E_1, E_2 are disjoint full continua in V and $A(E_1, E_2) := \widehat{\mathbb{C}} \setminus (E_1 \cup E_2)$.

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Define

$$\mathscr{D}_1(\phi)(z, w) = \left| \log \frac{|\phi'(z)\phi'(w)||z - w|^2}{|\phi(z) - \phi(w)|^2} \right|$$

for $(z, w) \in V \times V$, $z \neq w$, and define

$$\mathscr{D}_1(\phi, V) = \|\mathscr{D}_1(\phi)(z, w)\|_\infty.$$

Distortion

Lemma

Suppose that $\mathscr{D}(\phi, V) = \delta < \infty$. Then
(a) $\mathscr{D}_1(\phi, V) \leq 2\pi\delta$.
(b) Assume that V contains $0, \infty$ and $\mathbb{D}(1, r_0)$ for some $r_0 > 0$. If ϕ fixes $0, 1$ and ∞, then there exists a constant $C(r_0) > 0$ depending only on r_0 such that

$$\mathrm{dist}_s(\phi(z), z) \leq C(r_0)\delta.$$

Corollary

Let $\phi_n : V \to \hat{\mathbb{C}}$ be a sequence of univalent maps normalized by fixing three distinct points in V. Then ϕ_n converges uniformly to the identity if $\mathscr{D}(\phi_n, V) \to 0$ as $n \to \infty$.

Pullback system

Let g be a geometrically finite rational map with $\mathcal{F}_g \neq \emptyset$.
Let $X_0 \subset \widehat{\mathbb{C}}$ be a finite set with $g(X_0) = X_0$. Denote $X = \cup_{n\geq 0} g^{-n}(X_0)$.
For $x \in X$, denote by $n(x)$ the minimal integer $n \geq 0$ s.t. $g^n(x) \in X_0$.

Lemma

There exist disks $U_x \ni x$ for $x \in X$ satisfying the following conditions:
(a) $\overline{U_x} \setminus \{x\}$ contains no critical values of g.
(b) $\overline{U_x} \setminus \{x\}$ is disjoint from $\mathcal{P}_g$ if $n(x) \geq 1$.
(c) $\overline{U_x} \setminus \{x\}$ is disjoint from $g^{-n(x)}(X_0)$.
(d) U_x is a component of $g^{-1}(U_y)$ if $g(x) = y$ and $n(x) \geq 1$.
(e) $\max_{n(x)=n}\{\mathrm{diam}_s U_x\} \to 0$ as $n \to \infty$.

$\{U_x\}$ is called a **pullback system** at X.

Distortion control

Denote $U_x(r) = \chi_x^{-1}(\mathbb{D}(r))$ for $r \in (0,1)$, where $\chi_x : U_x \to \mathbb{D}$ is conformal with $\chi_x(x) = 0$.

Theorem [CT1]

Let $\{U_x\}_{x\in X}$ be a pullback system at X. Let V be an open set compactly contained in $\widehat{\mathbb{C}}\setminus\overline{X}$. Then there exist a constant $r_0 \in (0,1)$ and an increasing function $C(r)$ on $(0, r_0)$ with $C(r) \to 0$ as $r \to 0$, such that for any $r \in (0, r_0)$, if

$$\phi : \widehat{\mathbb{C}}\setminus \bigcup_{n(x)\le n} \overline{U_x(r)} \to \widehat{\mathbb{C}}$$

is univalent for some integer $n \ge 0$, then $\mathscr{D}(\phi, V) \le C(r)$.

Application

Let g be a geometrically finite rational map with $\mathcal{F}_g \neq \emptyset$.
Let $X_0 \subset \widehat{\mathbb{C}}$ be a finite set with $g(X_0) = X_0$. Denote $X = \cup_{n\geq 0} g^{-n}(X_0)$.
Let $\{U_x\}$ be a pullback system at X. Denote $\mathcal{U}_n(t) = \cup_{n(x)\leq n} \overline{U_x(t)}$.

Theorem

Let F_t be a semi-rational map with $\deg F_t = \deg g$ which has neither Thurston obstruction nor connecting arc. Suppose that there exists a univalent map $p_t : \widehat{\mathbb{C}} \smallsetminus (\mathcal{U}_0(t) \cap g^{-1}(\mathcal{U}_0(t))) \to \widehat{\mathbb{C}}$ s.t. the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} \smallsetminus g^{-1}(\mathcal{U}_0(t)) & \xrightarrow{p_t} & \widehat{\mathbb{C}} \\ g\downarrow & & \downarrow F_t \\ \widehat{\mathbb{C}} \smallsetminus \mathcal{U}_0(t) & \xrightarrow{p_t} & \widehat{\mathbb{C}} \end{array}$$

If F_t is holomorphic in $\widehat{\mathbb{C}} \smallsetminus p_t(\mathcal{U}_1(t))$, then F_t is c-equivalent to a rational map f_t which converges uniformly to g as $t \to 0$.

Application

- The pinching path is convergent under certain condition [CT1].
- An explicit estimation of the Riemann mapping $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathcal{M}$ near cusps or Misuriwz points [CT2].
- Construction of a stretching ray started from a cubic polynomial with a critical cycle and a parabolic cycle [AC].
- The twist sequence is convergent under certain condition [C].

Non-separating multi-annulus

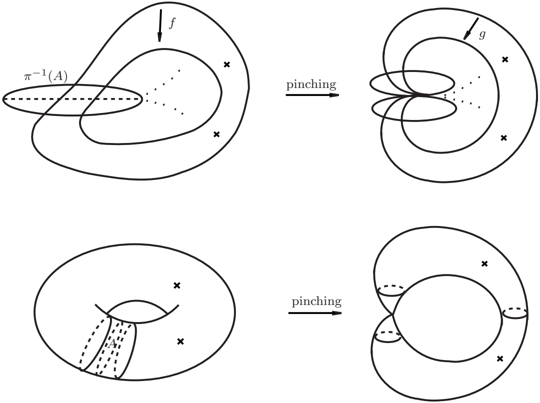
Let f be a rational map. Denote by $\mathcal{B}_f$ be the set of wandering points in attracting or parabolic basins whose orbits are disjoint from $\mathcal{P}_f$.
Define $z_1 \sim z_2$ if $f^n(z_1) = f^m(z_2)$ for integers $n, m > 0$. Then

$$\mathscr{R}_f = \mathcal{B}_f / \sim = \{\text{punctured torii}\} \cup \{\text{punctured cylinders}\}$$

is the **(punctured) quotient space** of f. Denote by $\pi_f : \mathcal{B}_f \to \mathscr{R}_f$ the natural projection.

A **multi-annulus** $\mathscr{A} \subset \mathscr{R}_f$ is a finite disjoint union of annuli whose boundaries are pairwise disjoint Jordan curves in $\mathscr{R}_f$ s.t. each component of $\pi_f^{-1}(\partial \mathscr{A})$ is an arc.
A multi-annulus $\mathscr{A} \subset \mathscr{R}_f$ is **non-separating** if for any choice of finitely many components of $\pi_f^{-1}(\mathscr{A})$, the union T of their closures does not separate the Julia set $\mathcal{J}_f$.

Pinching



Pinching of a rational map: from attracting to parabolic

Piching

Theorem [CT]

Let f be a geometrically finite rational map and $\mathscr{A} \subset \mathscr{R}_f$ be a non-separating multi-annulus. Let $f_t = \phi_t \circ f \circ \phi_t^{-1}$ $(t \geq 0)$ be the pinching path supported on $\mathscr{A}$. Then the following properties hold:
(a) f_t converges uniformly to a rational map g as $t \to \infty$.
(b) ϕ_t converges uniformly to a map φ of $\widehat{\mathbb{C}}$ as $t \to \infty$.
(c) $\varphi(\mathcal{J}_f) = \mathcal{J}_g$.
(d) $M[g] \subset \partial M[f]$.

Application to cubic polynomials

Denote by $\mathcal{S}_n$ the space of cubic polynomials with the form

$$P_{a,b}(z) := z^3 - 3a^2 z + b,$$

such that the critical point a has exact period n.

Theorem [AC]

Let $P \in \mathcal{S}_n$ be a cubic polynomial with a parabolic cycle X. Let R_θ be an external ray landing on a point of X such that R_θ is adjacent to the parabolic basin containing $-a$. Then there exists a stretching ray $\{P_t\}$ in an escaping component of $\mathcal{S}_n$ landing at P such that the Böttcher coordinate of the escaping critical point $-a$ has argument θ.

Twist deformation

Let f be a geometrically finite rational map and $\mathscr{A} \subset \mathscr{R}_f$ be a non-separating multi-annulus.
Let $k(A) \neq 0$ be an integer for each annulus A of $\mathscr{A}$. Define a repeated Dehn twist on each annulus A with times $n \cdot k(A)$ for $n \geq 1$.
These repeated Dehn twists induce a quasiconformal deformation of f as

$$f_n = \phi_n \circ f \circ \phi_n^{-1}.$$

Theorem [C]

The sequence $\{f_n\}$ converges uniformly to a geometrically finite rational map g as $t \to \infty$.

Remark. In general, $\mathcal{J}_{f_n} \not\to \mathcal{J}_g$ and $M[g] \not\subset \partial M[f]$.

Ideas of the proof

$$(f, \mathscr{A}) \xRightarrow{\text{pinching}} h \xRightarrow{\text{plumbing}} g_0 \xRightarrow{\text{q.c}} g \xRightarrow{\text{implosion}} f$$

- The combinatorics of the plumbing depends on the structure of multi-annulus and the twist number.
- The analytic structures of f and g is related by a Lavaurs map.
- We will do a parabolic implosion surgery by this Lavaurs map.
- The existence and uniqueness of the map g is due to the Teichmuller contraction.

Parabolic implosion surgery

Let (g, x) be a local parabolic fixed point.

- Let $\mathcal{V}, \mathcal{V}'$ be the union of some attracting petals and repelling petals bounded by vertical Fatou line, respectively, s.t. they are disjoint.
- Cut the closures of these petals and glue the boundaries by a translation under the Fatou coordinates. We obtained a punctured sphere.
- The map g induces a holomorphic map F.
- Assume F has an attracting fixed point. Replace the attracting petals by their images, the resulting holomorphic map is a twist of the original one.

Iteration in Teichmuller spaces

- Assume that f has an attracting domain U where the twist defined. Then there exists a translation L from the attracting cylinders to repelling cylinders of g s.t. near one end, its quotient space under $\Lambda \circ L$ (Λ is the horn map) is holomorphically isomorphic to the quotient space of U under f.
- To find the analytic structure of g, we define a map from the Teichmuller space of the attracting cylinders to itself.
- The contraction of the map on the Teichmuller space is guaranteed by the distortion control.

Thank you for your attention!