On the pullback relation on curves induced by a Thurston map

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Abstract

Via taking connected components of preimages, a Thurston map $f : (S^2, P_f) \to (S^2, P_f)$ induces a pullback relation on the set of isotopy classes of curves in the complement of its postcritical set P_f. We survey known results about the dynamics of this relation, and pose some questions.

1 Introduction

An orientation-preserving branched covering $f : S^2 \to S^2$ of degree at least two is a *Thurston map* if its *postcritical set* $P_f = \cup_{n>0} f^n(C_f)$ is finite, where C_f is the finite set of branch (critical) points at which f fails to be locally injective.

A fundamental theorem in complex dynamics–Thurston's Characterization and Rigidity Theorem [DH]–asserts that apart from a well-known and ubiquitous set of counterexamples, the dynamics of rational Thurston maps is determined, up to holomorphic conjugacy, by its conjugacy-up-to-isotopy-relative-to-P_f class.

The set of isotopy classes relative to P_f of Thurston maps g for which $P_g = P_f$ admits the structure of a countable semigroup under composition. Its elements are akin to elements of the mapping class group of a surface with a finite nonempty set of marked points. This perspective is extremely useful in developing intuition for the range of potential behavior of and structure theory for Thurston maps. The mapping class group of a surface, and hence its individual elements, acts naturally on the countably infinite set of isotopy classes of curves on the surface. It is natural to try to do something similar for Thurston maps.

Since the set P_f contains the branch values of f, the restriction $f : S^2 - f^{-1}(P_f) \to S^2 - P_f$ is a covering map. It follows that a component $\widetilde{\gamma}$ of the inverse image $f^{-1}(\gamma)$ of a simple closed curve γ in $S^2 - P_f$ is a simple closed curve in $S^2 - f^{-1}(P_f)$. Since P_f is forward-invariant, we have an inclusion $S^2 - f^{-1}(P_f) \hookrightarrow S^2 - P_f$, so the curve $\widetilde{\gamma}$ is again a simple closed curve in $S^2 - P_f$. Abusing terminology, we'll call $\widetilde{\gamma}$ a *preimage* of γ, or sometimes say γ *lifts*, or *pulls back*, to $\widetilde{\gamma}$. By lifting isotopies, we obtain a pullback relation $\stackrel{f}{\leftarrow}$ on the set of simple closed curves $\mathcal{C}$ up to isotopy. The curve γ might have several preimages, so we obtain an induced relation instead of a function. A preimage of an inessential curve is again inessential. Similarly, a preimage of a peripheral curve–one which is isotopic into any small neighborhood of a single point in P_f–is either again peripheral, or is inessential. We call inessential and peripheral curves *trivial*, and note that these are invariant under the pullback relation.

When $\#P_f = 4$, the pullback relation induces–almost–a function on the set of nontrivial curves. On the one hand, distinct nontrivial curves in this case must intersect. On the other hand, distinct components of $f^{-1}(\gamma)$ are in general disjoint. It follows that there can be at most one class of nontrivial preimage, and we almost get a function in this case. Why "almost"? Typical examples have the property that for some curve, each of its preimages are trivial. So while the mapping class group acts naturally on e.g. the infinite diameter curve complex, it is less clear how to construct a nice complex related to curves on which a Thurston map acts via pullback. This relative lack of preserved structure makes answering even basic questions challenging.

In this note, I survey some known results about the dynamical behavior of taking iterated preimages of curves under a given Thurston map.

2 Conventions and notation

Throughout, f denotes a Thurston map, P its postcritical set, and d its degree. Unless otherwise stated, f has hyperbolic orbifold and $\#P \geq 4$. We denote by

- $\mathcal{C}$, the countably infinite set of istopy classes of unoriented, essential, simple, nonperipheral curves in $S^2 - P$ (we will often call such elements simply "curves", abusing terminology);
- o, the union of the isotopy classes of inessential and peripheral curves, i.e. the trivial ones;
- $\overline{\mathcal{C}} := \mathcal{C} \cup \{o\}$;
- $\stackrel{f}{\leftarrow}$, the pullback relation on $\overline{\mathcal{C}}$ induced by $\gamma \mapsto \delta \subset f^{-1}(\gamma)$, where $[\gamma] \in \overline{\mathcal{C}}$ and δ is a component of $f^{-1}(\gamma)$;
- $m\mathcal{C}$, the set of multicurves Γ, defined as (necessarily finite) subsets of $\mathcal{C}$ represented by pairwise disjoint curves distinct up to isotopy;
- $f^{-1} : m\mathcal{C} \to m\mathcal{C}$ the function induced by pullback;
- $\overline{\mathcal{A}}$ and $\mathcal{A}$, the set of curves contained in cycles of $\stackrel{f}{\leftarrow}$ in $\overline{\mathcal{C}}$ and $\mathcal{C}$, respectively;
- $\mathcal{W} \subset \mathcal{C}$, the set of "wandering" curves γ_0, namely, those for which there is an infinite sequence $\gamma_n, n \geq 0$, of distinct nontrivial curves satisfying $\gamma_n \stackrel{f}{\leftarrow} \gamma_{n+1}, n \geq 0$;
- the relation $\stackrel{f}{\leftarrow}$ has a *finite global attractor* if $\mathcal{W}$ is empty and $\mathcal{A}$ is finite;
- $\iota(\alpha, \beta)$, the unsigned intersection number between two elements of $\mathcal{C}$;
- $\mathrm{Teich}(S^2, P)$, the Teichmüller space of the sphere marked at the set P;
- $\sigma_f : \mathrm{Teich}(S^2, P) \to \mathrm{Teich}(S^2, P)$, the holomorphic self-map obtained by pulling back complex structures.

3 Non-dynamical properties of $\stackrel{f}{\leftarrow}$.

3.1 Known general results

Thinking non-dynamically first, we have the following known results about the pullback relation $\stackrel{f}{\leftarrow}$.

1. Each nonempty fiber is dense in the Thurston boundary; in particular, each nontrivial fiber is infinite [KPS]. Here, by the fiber over β, we mean $\{\alpha : \alpha \stackrel{f}{\leftarrow} \beta\}$.
2. The relation $\stackrel{f}{\leftarrow}$ can be trivial in the sense that the only pairs are of the form $\gamma \stackrel{f}{\leftarrow} o$. Equivalently, σ_f is constant. See [KPS], correcting an argument appearing originally in [BEKP].

3. The relation $\stackrel{f}{\leftarrow}$ satisfies a Lipschitz-type inequality related to intersection numbers: $\iota(\widetilde{\alpha}, \widetilde{\beta}) \leq d \cdot \iota(\alpha, \beta)$ whenever $\alpha \stackrel{f}{\leftarrow} \widetilde{\alpha}, \beta \stackrel{f}{\leftarrow} \widetilde{\beta}$.
4. Multicurves are in natural bijective correspondence with boundary strata in the augmented Teichmüller space, which is known to be the completion of Teichmüller space in the Weil-Petersson (WP) metric. A result of Selinger [Sel] shows that $\sigma_f : \mathrm{Teich}(S^2, A) \to \mathrm{Teich}(S^2, B)$ extends to the WP completion, sending the stratum corresponding to a multicurve Γ to the stratum corresponding to the multicurve $f^{-1}(\Gamma)$. It follows that analytical tools for studying σ_f can be used to study properties of the combinatorial relation $\stackrel{f}{\leftarrow}$ [Pil2], [KPS].
5. Multicurves are in natural bijective correspondence with certain abelian subgroups of the mapping class group $\mathrm{Mod}(S^2, P_f)$. The pullback function can be encoded using the associated induced virtual endomorphism on the mapping class group $\phi_f : \mathrm{Mod}(S^2, P_f) \dashrightarrow \mathrm{Mod}(S^2, P_f)$. It follows that algebraic tools can be used to study properties of the combinatorial relations $\stackrel{f}{\leftarrow}$ and f^{-1} on $\mathcal{C}$; see [Pil2] and [KL].

Question 3.1 *If the pullback relation $\stackrel{f}{\leftarrow}$ is not trivial, must it be surjective?*

It seems very likely that the answer is no, for the following reason. The Composition Trick should allow one to build examples where the image of σ_f has positive dimension and codimension, so that its image misses many strata. ‘

3.2 Mechanisms for triviality of $\stackrel{f}{\leftarrow}$

There seem to be three or four mechanisms via which $\stackrel{f}{\leftarrow}$ can be trivial.

1. **Composition trick.** The map $f : (S^2, A) \to (S^2, B)$ may factor through (S^2, C) with $\#C = 3$ (C. McMullen, [BEKP]). Even if one is interested in dynamics $f : (S^2, P) \to (S^2, P)$, the possibility that f factors as $(S^2, P) \to (S^2, C) \to (S^2, P)$ makes investigating non-dynamical properties of maps of the form $(S^2, A) \to (S^2, B)$ very natural.
2. **NET maps.** A. Saenz [Mal] found an example of a Thurston map f for which σ_f is constant but for which f does not decompose as in the Composition Trick. Here is his example, from a different point of view.

   Let E be an elliptic curve over $\mathbb{C}$. There are 8 distinct points of order 3; under the involution $z \mapsto -z$ these 8 points descend to a set of 4 points A on $\mathbb{P}^1$ whose cross-ratio is constant as E varies. Now take E to be the square torus and let $f = \mathbb{P}^1 = E/\pm 1 \to E/\pm 1 = \mathbb{P}^1$ is the degree 9 "tripling" flexible Lattès map, $B =$ the corner points of $E/\pm 1$, $A =$as above. Then σ_f is constant and so $\stackrel{f}{\leftarrow}$ is trivial. One can see this triviality directly by observing that the action of $\mathrm{PSL}_2(\mathbb{Z})$ is transitive on curves (since it acts transitively on extended rationals regarded as slopes), that A is invariant under this action (since points of order 3 are invariant under group-theoretic automorphisms), and that the horizontal curve has all preimages inessential or peripheral (as a single easy picture shows).
3. **Sporadic examples.** Let f be the unique (up to pre- and post-composition by independent automorphisms) degree four rational map with three double critical points mapping to necessarily distinct critical values (v_1, v_2, v_3). Let $B = \{v_1, v_2, v_3, w\}$ and $A = R^{-1}(w) = (z_1, z_2, z_3, z_4)$. Then the cross-ratio of the z_i's is constant in w, whence σ_f is constant and so $\stackrel{f}{\leftarrow}$ is trivial. To see this, note that as $w \to v_1$, the fiber over w breaks up into two subsets, one subset of three points converging to z_1, and another point converging to a point z_1' distinct from z_1. Normalizing so $z_1 = 0$ and $z_1' = \infty$ and scaling

via multiplication with a nonzero complex constant shows that the cluster of three points in the fiber approach the cube roots of unity.

4. **Combinations of the above.**

Question 3.2 *Do there exist examples f with σ_f constant and* $\deg(f)$ *prime?*

G. Cui (CAS) has thought about the general case, see [Cui]. It is natural to look for the simplest such examples. By cutting along a maximal multicurve, one may restrict to the case $\#P = 4$; let's call these "minimal". It is natural to look for examples which do not factor as in the Composition Trick; let's call these "primitive".

Question 3.3 *What are the minimal primitive branched covers $f : (S^2, A) \to (S^2, B)$ for which $\stackrel{f}{\leftarrow}$ is trivial?*

3.3 Computation of $\stackrel{f}{\leftarrow}$

Though the set of curves $\mathcal{C}$ is complicated, it is conveniently described by a variety of coordinate systems, e.g. by train tracks and by recording intersection numbers with edges in a fixed triangulation with vertex set P_f. But expressing the pullback relation in these coordinates can be very complicated: while pulling back from $S^2 - P_f$ to $S^2 - f^{-1}(P_f)$ is easy (just lift intersection numbers), the "erasing map" is hard to write down in closed form and leads to continued-fraction-like cases.

When $\#P_f = 4$, though, the set of curves $\mathcal{C}$ can be encoded by "slopes" in the extended rationals $\mathbb{Q} \cup \{1/0\}$, the pullback relation is a function, and things are a bit easier but are still quite complicated.

If f is a so-called *Nearly Euclidean* Thurston map (NET map) (see [CFPP]), there is an algorithm that computes the image of a slope under pullback. This can be done easily by hand, and has been implemented. NET maps can be easily encoded by combinatorial input. W. Parry has written a computer program that implements this algorithm. The website `http://www.math.vt.edu/netmaps/index.php`, maintained by W. Floyd, contains a database of tens of thousands of examples. For NET maps, it appears that this ability to calculate the pullback map on curves (and related invariants, such as the degree by which preimages map, and how many preimages there are) leads to an effective algorithm for determining whether a given example is, or is not, equivalent to a rational map.

When $\#P_f = 4$ and f is the subdivision map of a subdivision rule of the square pillowcase (like in Figure 6, W. Parry has written a program for computing the image of a slope under pullback (personal communication).

Question 3.4 *Are there any settings in which one can effectively compute $\stackrel{f}{\leftarrow}$ when $\#P_f \geq 5$?*

3.4 When each curve has a nontrivial preimage

The example studied by Lodge [Lod] is, nondynamically speaking, the generic cubic: four simple critical points mapping to four distinct critical values. Nondynamically speaking, such a map is unique. In this example, each nontrivial curve has a nontrivial preimage.

Question 3.5 *Suppose $f : (S^2, A) \to (S^2, B)$ is generic in the sense that A consists of $2d - 2$ simple critical points mapping to a set B of $2d - 2$ distinct critical values. Does each nontrivial curve in $S^2 - B$ have a nontrivial preimage?*

Up to pre- and post-composition with homeomorphisms there is a unique such map [BE]. If each γ has a nontrivial preimage, so does $h(\gamma)$, where $h : (S^2, P_f) \to (S^2, P_f)$ is a homeomorphism that lifts under f. Since the set of such h is a finite-index subgroup of $\mathrm{Mod}(S^2, P_f)$, checking this is a finite computation.

4 Dynamical properties

Here we present examples and known results about the possible dynamical behavior of $\stackrel{f}{\leftarrow}$.

1. **Example:** *Every curve iterates to the trivial curve.* This happens for z^2+i. Here is one way to see this. Examining the possibilities for how the bounded region enclosed by a curve meets the finite postcritical set $\{i, i-1, -i\}$, one sees that a curve must eventually become trivial unless it surrounds both $-i$ and $i-1$. For this type of curve α, there is at most one nontrivial curve β with $\alpha \stackrel{f}{\leftarrow} \beta$ and β a curve of the same type. Moreover, $\deg(\alpha \stackrel{f}{\leftarrow} \beta) = 1$. Equipping the complement of the postcritical set with the hyperbolic metric, the Schwarz Lemma shows that the length of a geodesic representative of β is strictly shorter than that of α. Iterating this process, it follows that such a curve cannot be periodic under $\stackrel{f}{\leftarrow}$: points in its orbit cannot get too complicated, since otherwise they would have to get long, so they must eventually become a different type of curve and thus become trivial upon further iteration.

   The "airplane" quadratic polynomial $f(z) = z^2 + c$, with the origin periodic of period 3 and $\mathrm{Im}(c) = 0$, is another example [KL].

2. **Question 4.1** *Does there exist an example of a Thurston map f for which the pullback relation induced by f is nontrivial but that induced by some iterate f^n is trivial?*

3. **Theorem.** If f is rational and non-Lattès, $\mathcal{A}$ must be finite [Pil2]. The proof uses the decomposition theory.

4. **Conjecture:** *If f is rational and not a flexible Lattès example then the pullback relation $\stackrel{f}{\leftarrow}$ has a finite global attractor.*

   There is partial progress on this conjecture.

   (a) Kelsey and Lodge [KL] verify this for all quadratic non-Lattès maps with four postcritical points.

   (b) (Dudko-P., unpublished) If f is a critically fixed rational map, α an edge in the planar connected multigraph $\mathcal{G}$ that describes f via the blowing up construction [CGN+], $\gamma \stackrel{f}{\leftarrow} \widetilde{\gamma}$, and $\iota(\gamma, \alpha) > 0$, then it is easy to see that unless γ is homotopic to the boundary of a regular neighborhood of an edge-path homeomorphic to an embedded arc in $\mathcal{G}$ and $\iota(\gamma, \alpha) = 1$, we must have $\iota(\widetilde{\gamma}, \alpha) < \iota(\gamma, \alpha)$. Thus the global attractor $\mathcal{A}$ consists of such γ.

   (c) If the virtual endomorphism ϕ_f on the mapping class group is contracting, then $\stackrel{f}{\leftarrow}$ has a finite global attractor [KPS, Thm. 7.2].

   (d) If the correspondence on moduli space (in the direction of σ_f) has an nonempty invariant compact subset, then ϕ_f is contracting, so there is a finite global attractor. If moduli space admits an incomplete metric which is (i) uniformly contracted by σ_f, and (ii) whose completion is homeomorphic to that of the WP metric, then the trivial curve is a finite global attractor [KPS, Thm. 7.2]. The latter occurs for $f(z) = z^2+i$; the correspondence on moduli space is the inverse of a Lattès map with three postcritical points and Julia set the whole sphere, which expands the Euclidean orbifold metric.

   Bounds on the size of the attractor. Since up to conjugacy there are only finitely many non-flexible Lattès rational maps with a given degree and size of postcritical set, we have $\#\mathcal{A}_f \leq C(\deg f, \#P_f)$. I know very little about the behavior of this function.

   Fix a degree $d \geq 2$.

(a) Certainly $\#\mathcal{A}$ can be large if P_f is large, e.g. for renormalizable quadratic polynomials. Other examples can be constructed by perturbing flexible Lattès examples. One can find hyperbolic sets consisting of invariant curves which are stable under perturbation; a result of X. Buff and T. Gauthier [BG, Cor. 3] implies that such maps are limits of sequences of pcf hyperbolic maps with the maximum number $2d-2$ attracting cycles.

In composite degrees, $\#\mathcal{A}$ can be small (say zero), by taking e.g. examples with σ_f constant. Using McMullen's compositional trick and Belyi functions one can easily build both hyperbolic rational maps and rational maps with Julia set the whole sphere having the property that σ_f constant and $\#P_f$ arbitrarily large.

(b) Results of G. Kelsey and R. Lodge [KL] show that for quadratic rational maps f with $\#P_f = 4$, we have $\#\mathcal{A} \leq 4$.

The bound might be explained as follows. The map f corresponds (not quite bijectively) to a repelling fixed-point p of a correspondence $g = Y \circ X^{-1}$ on moduli space. In the nonexceptional cases, this is actually a rational map $g : \mathbb{P}^1 \to \mathbb{P}^1$. There appears to be a natural bijection between invariant (multi)curves for f and periodic internal rays joining points in periodic superattracting cycles of g (these lie at infinity in moduli space) to p. I've confirmed this also for critically fixed polynomials with three finite critical points.

Question 4.2 *What is the relationship between the existence of invariant multicurves (and, more generally, of periodic curves) in the dynamical plane of a rational Thurston map f, and the dynamics of the induced correspondence on moduli space related to the fixed-point of the correspondence given by f?*

For quadratics with four postcritical points, the analysis of [KL] seems to confirm the intuition that periodic curves in the dynamical plane are related to accesses landing at the associated fixed-point that are periodic under the correspondence, in this case the inverse of a critically finite rational map with three postcritical points. But in higher degrees with $\#P_f = 4$, the correspondence need not the inverse of such a map, and the situation is more complicated; see e.g. [Lod].

In higher dimensions, another first natural example to try is the case of f a critically fixed polynomial with four finite simple critical points. One would need to show the existence of internal rays in two complex dimensions. This example is beautifully symmetric and possesses many invariant lines that might make the problem more tractable.

(c) D. Margalit and R. Winarski (personal communication) have results for polynomials, using combinatorial techniques and an analysis of an induced map on the arc complex. They exploit the fact that if α is an arc joining a finite postcritical point $q = f(p)$ to the point at infinity, and if $\widetilde{\alpha}$ is a lift of α based at p, then $\widetilde{\alpha}$ joins p to the point at infinity, and $f|\widetilde{\alpha}$ is a homeomorphism. This implies that intersection numbers of curves with such arcs cannot grow under pullback.

5. **Examples with symmetries.** In the search for other examples, it is natural to consider $\mathrm{Mod}(f) = \{h : hf \simeq fh \text{ rel } P_f\}$; here $\simeq$ denotes isotopy. We recall four facts:

(a) If we work with pure mapping classes, $P\mathrm{Mod}(f)$ has no elements of finite order.

(b) If f is rational, $P\mathrm{Mod}(f)$ is trivial, unless f is a flexible Lattès example, in which case it is the group $P\Gamma(2)$ which is free on two generators.

(c) If f is obstructed, and there is an invariant multicurve with 1 as some eigenvalue, then the multitwist about such a multicurve with powers given by the eigenvector gives an element of $\mathrm{Mod}(f)$ [Pil1].

(d) Thurston maps are like mapping classes. If f is obstructed, there is a canonical decomposition by cutting along a certain invariant multicurve. The "pieces" might contain cycles of degree one: mapping class elements, each with its own centralizer. The fact that the decomposition is canonical means that the centralizers of the pieces will embed into $\mathrm{Mod}(f)$. Using this idea one can create examples of Thurston maps with a variety of prescribed behaviors: $\mathcal{A}$ infinite and $\mathcal{W}/\stackrel{f}{\leftarrow}$ infinite, for example: just find one piece on which the map is the identity, and another on which it is a pseudo-Anosov map.

- Such examples occur among critically fixed maps: just blow up e.g. 5 disjoint arcs, and note that the resulting map f is isotopic to the identity on their complement, which supports a partially pseudo-Anosov map. This creates other examples with $\mathrm{Mod}(f)$ containing (at least) products of centralizers of prescribed mapping class elements.
- L. Bartholdi and D. Dudko give an explicit example of f with $\mathrm{Mod}(f)$ infinitely generated [BD].

6. **Expanding vs. nonexpanding maps.** The examples in 5(d) are not isotopic to expanding maps: Levy cycles–cycles in which each curve maps by degree 1–are obstructions. However, there exist expanding maps with rich dynamical behavior of the pullback relation.

- Blow up the 2x2 Lattès doubling example along the middle upper vertical edge to get a Thurston map f; see Figure 6. The "rim" of the square pillowcase–the common boundary of the two squares at left– is an invariant Jordan curve containing P_f: that is, f is the subdivision map of a finite subdivision rule. The map f is isotopic to an expanding map. To see this, one may apply either the characterization in either [CFP] or [BM]. The vertical curve is an obstruction with multiplier 1, and the horizontal curve is invariant. Let T be the Dehn twist about this vertical curve, so that $fT = Tf$. This immediately implies (i) $\mathcal{A}_f$ is infinite, since the orbit of the horizontal curve under T will consist of f-invariant curves, and (ii) if we put $g = Tf$, then the horizontal curve wanders. To see this, note that $g^{-n}(\gamma) = (Tf)^{-n}(\gamma) = T^{-n}f^{-n}(\gamma) = T^{-n}\gamma$ as required. I do not know if g is expanding, however. I would guess that $g_N := Tf^N$ is expanding for sufficiently large N, and one could use this instead to build a similar example.
- **Question 4.3** *Suppose* $\mathrm{Mod}(f)$ *is trivial. Could* $\mathcal{A}_f$ *be infinite? Could* $\mathcal{W}_f$ *be nonempty?*

7. **Examples of complete global picture of dynamics.**

(a) As mentioned above, for rational critically fixed maps f, we have a good qualitative picture of the global dynamics induced by pullback on curves. The only periodic curves are fixed curves; these comprise boundaries of small regular neighborhoods of embedded arc edge-paths in the multigraph $\mathcal{G}$ defining f via blowing up surgery; any other curve under pullback either becomes trivial or one of these fixed curves.

(b) In [FKK$^+$, Theorem 8], it is shown that if g is an obstructed quadratic NET map with exactly one critical postcritical point, then every slope eventually iterates to either the trivial curve, or the obstruction, or to a finite family of wandering curves, which might be empty; all cases occur. Thus in this case we have a complete global description of the dynamics of $\stackrel{f}{\leftarrow}$. More generally, for other obstructed quadratics with four postcritical points, it seems quite likely that the analysis in [KL] yields a similar explicit answer. These examples are not expanding, however.

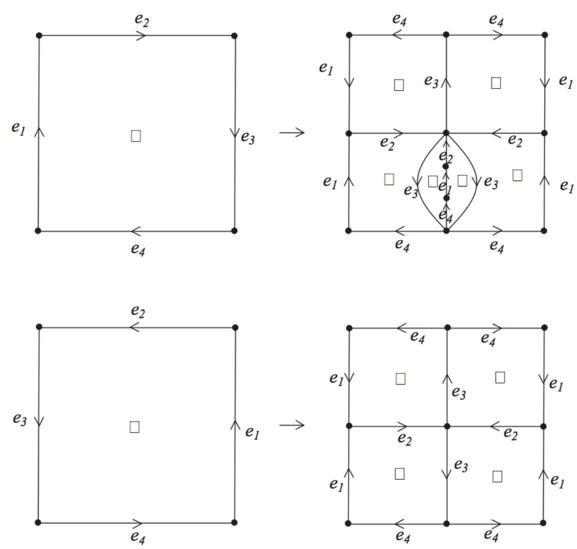

Figure 1: The codomain is the union of the two squares at left along their boundaries as indicated to form a square "pillowcase". Each square at right is identified with the square just to its left by a translation, so that the pillowcases are identified. The figure shows a cell structure in domain and codomain. The map f goes in the opposite direction to the indicated arrow and defines a cellular degree 5 map from the pillowcase to itself. The four corners of the pillowcase form the postcritical set. *Figure by W. Floyd.*

(c) **Question 4.4** *Is there an example of an expanding obstructed Thurston map f for which one has a complete description of the global dynamics of $\xleftarrow{f}$?*
The blown-up Lattès example of Figure 6 would be natural candidates for analysis. Other tractable candidates might be found among the class of Nearly Euclidean Thurston maps; one would need a mechanism for finding expanding NET maps.

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