THE POLYNOMIALS ASSOCIATED WITH A JULIA SET

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ABSTRACT

We prove that, with two exceptions, the set of polynomials with Julia set $\mathscr{J}$ has the form $\{\sigma p^n : n \in \mathbb{N}, \sigma \in \Sigma\}$ where p is one of these polynomials and Σ is the symmetry group of $\mathscr{J}$. The exceptions occur when $\mathscr{J}$ is a circle or a straight line segment.

Several papers [**1, 2, 3, 5**] have appeared dealing with the relation between polynomials having the same Julia set $\mathscr{J}$ (for notation the reader is referred to [**8**]). This relation is very simple and by no means surprising.

THEOREM. *For any Julia set (of a polynomial) $\mathscr{J}$, which is not a circle or a straight line segment, there exists a polynomial p such that any polynomial with Julia set $\mathscr{J}$ can be written in the form σp^n, where σ is a rotation mapping $\mathscr{J}$ onto itself, and n is a positive integer.*

Proof. We denote by $\Sigma = \Sigma(\mathscr{J})$ the symmetry group of $\mathscr{J}$, that is, the group of Möbius transformations mapping $\mathscr{J}$ onto itself. There exists a polynomial p of lowest degree associated with $\mathscr{J}$. Without loss of generality, we may assume that p is given by

$$p(z) = z^d + a_{d-2} z^{d-2} + \ldots + a_0,$$

since otherwise we could consider the Julia set $M(\mathscr{J})$, the conjugate group $M\Sigma M^{-1}$ and the conjugate polynomial $M \circ p \circ M^{-1}$, where $M(z) = \alpha z + \beta$ is a suitably chosen Möbius transformation. Baker and Eremenko [**1**] have shown that p may be written as

$$p(z) = z^\mu p_0(z^m),$$

where m is the order of the symmetry group of $\mathscr{J}$, namely

$$\Sigma = \{z \longmapsto \delta z : \delta^m = 1\},$$

and p_0 is a polynomial. In a similar way, any polynomial q associated with $\mathscr{J}$ can be written in this form, say $q(z) = z^\nu q_0(z^m)$. It is easily seen that $p \circ q = \delta q \circ p$ holds, and also that the polynomials

$$\hat{p}(z) = z^\mu (p_0(z))^m \quad \text{and} \quad \hat{q}(z) = z^\nu (q_0(z))^m$$

commute; this can also be found in [**1**]. According to results due to Julia [**6**], Fatou [**4**] and Ritt [**7**], we have to consider three different cases.

(a) $\hat{p}(z) = z^d$ is a monomial. Then p is a monomial too, and the Julia set is the unit circle.

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(b) $\hat{p}$ is conjugate to some polynomial T, such that T or $-T$ is a Tchebychev polynomial for the interval $[-2,2]$. Then from

$$T(\alpha z+\beta) = \alpha z^{\mu}(p_0(z))^m+\beta$$

and the fact that T' has only simple zeros, it follows that $m \leqslant 2$, and hence $m = 2$. Also, we have $\beta = \pm 2$, since only the values ± 2 are critical for T, and so

$$T(\alpha z^2 \pm 2) = \alpha(p(z))^2 \pm 2$$

holds. We consider the respective conjugates

$$h(i\sqrt{\alpha}\, z) = i\sqrt{\alpha}\, p(z), \quad h(\sqrt{\alpha}\, z) = \sqrt{\alpha}\, p(z),$$

and obtain

$$(h(z))^2 = 2-T(2-z^2), \quad (h(z))^2 = 2+T(z^2-2),$$

respectively. In both cases, the interval $[-2,2]$ is completely invariant under h, and so the Julia set of h coincides with $[-2,2]$ and the Julia set of p is a straight line segment.

(c) There exist integers k and l such that $\hat{p}^k = \hat{q}^l$. In particular, it follows that $(\deg \hat{p})^k = (\deg \hat{q})^l$ and so d divides $\deg q$, that is, $q(z) = cz^{ds}+\ldots$. Now we use a device that can be found in Ritt's paper [7]. Let

$$B(z) = z\left(1+\frac{\alpha_1}{z^m}+\ldots\right)$$

be the normalized solution of Böttcher's functional equation

$$B(p(z)) = (B(z))^d;$$

it depends only on the Julia set $\mathcal{J}$, since, near infinity, $\log|B(z)|$ represents the Green's function of the outer domain of $\mathcal{J}$ with pole at ∞. In a neighbourhood of this point, we define a function r by the relation

$$B(r(z)) = c(B(z))^s.$$

Then $B(r(p(z))) = c(B(z))^{ds} = B(q(z))$, and hence

$$r \circ p = q$$

follows. Writing $r(z) = r_0(z)+O(1/z)$ as $z \to \infty$, where r_0, a polynomial, denotes the principal part of r at ∞, we obtain

$$q(z)-r_0(p(z)) = O(z^{-d}),$$

and hence $r = r_0$ is a polynomial. It is obvious that r maps each of the sets $\mathcal{J}$ and $\mathcal{F}$—the common Fatou set—onto itself, and hence either r has Julia set $\mathcal{J}$ or else $r(z) = \delta z$, $\delta^m = 1$, holds. Repetition of this argument, if necessary, leads to the conclusion that $q = \delta p^n$. Conversely, each polynomial δp^n has Julia set $\mathcal{J}$. This proves our theorem.

Remark. It is clear that in the non-exceptional case, δp^k and εp^n, $\varepsilon^m = 1$, are permutable if and only if $\delta^{\nu k-1} = \varepsilon^{\mu n-1}$ holds. This was already observed by Ritt.

References

1. I. N. Baker and A. Eremenko, 'A problem on Julia sets', *Ann. Acad. Sci. Fenn. Ser. A* 12 (1987) 229–236.

2. A. F. Beardon, 'Symmetries of Julia sets', *Bull. London Math. Soc.* 22 (1990) 576–582.
3. A. F. Beardon, 'Polynomials with identical Julia sets', *Complex Variables* 17 (1992) 195–200.
4. P. Fatou, 'Sur l'iteration analytique et les substitutions permutables', *J. Math.* 2 (1923) 343–362.
5. J. L. Fernández, 'A note on the Julia set of polynomials', *Complex Variables* 12 (1989) 83–85.
6. G. Julia, 'Mémoire sur la permutabilité des fractions rationelles', *Ann. Sci. École Norm. Sup.* 39 (1922) 131–215.
7. J. F. Ritt, 'On the iteration of rational functions', *Trans. Amer. Math. Soc.* 21 (1920) 348–356.
8. N. Steinmetz, *Rational iteration* (de Gruyter, Berlin, 1993).

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