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On the Bifurcation of Maps of the Interval*

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1. Introduction

In the last few years there has been considerable interest in the asymptotic behavior of maps of the interval into itself under iteration. Some of this interest has come from the theory of dynamical systems (where most authors have studied maps of the circle), and some of the interest has been generated by population biology. In population biology, maps of the unit interval have been used as models for the dynamics of populations with discrete generations.

One of the questions of most interest in the theory has been the determination of the limit sets of points for a map $f: I \to I$. Here $I=[0,1]$. The limit set of $x \in I$ is the set of limit points of the sequence $\{f^i(x)\}$. The superscript denotes repeated composition. Of particular interest are *periodic orbits*: points x such that $f^i(x)=x$ for some $i>0$. Even greater interest focuses upon attracting periodic orbits: if $f^i(x)=x$, then x lies in an *attracting* periodic orbit if $\left|\frac{d}{dx}(f^i)(x)\right|<1$. It is easily proved that there is a neighborhood of an attracting periodic orbit consisting of points which tend to the periodic orbit under iteration.

If the map $f: I \to I$ depends upon a parameter $\mu \in I$ (we write $f_\mu(x)$ or $f(x, \mu)$ and say that f is a one parameter *family* of maps.), then one is interested in the change of the limits sets as the parameter is varied. Vaguely, when there is a qualitative change, one says that μ is a *bifurcation* value of the parameter. One of the fundamental problems of bifurcation theory has been the description of those qualitative changes which are "generic" or "non-degenerate" in some suitable sense. This has been a *local* theory with regard to the bifurcation parameter: it studies changes near a particular bifurcation value. For a certain class of families of maps of the interval, we shall consider questions of the *global* bifurcation behavior. In particular, we shall describe a sequence of bifurcations which take place in a particular order with respect to the parameter. We shall see that for a large class of families of maps, the order in which periodic attractors appear and disappear is independent of the member of the family. This order can be computed explicitly.

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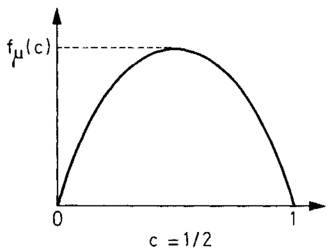

Fig. 1

The families we consider in most detail have the following properties:

(1) f_μ is a smooth function of both x and μ (at least C^3),
(2) $f_\mu(0)=f_\mu(1)=0$,
(3) f_μ has a single critical point c_μ in $I(\mu>0)$,
(4) f_0 is a constant function and $f_1(c_1)=1$.

Our initial arguments will also rely upon additional hypotheses which are later shown to be unnecessary. One of the most frequently studied families of maps of the interval has been the family of quadratic maps given by the equation $f_\mu(x)=4\mu x(1-x)$. The graph of this map is shown in Figure 1.
Thinking of this quadratic case, we indicate the nature of the results we obtain. Precise statements are made below. For the family of quadratic maps, there is at most one attracting periodic orbit for each value of μ [2]. Assign to μ the period $n(\mu)$ of this attracting periodic orbit, if there is one. Our main result gives an algorithm for computing the order of the numbers $n(\mu)$ for n less than an arbitrary bound. The surprising aspect of our results is that the "sequence" $n(\mu)$ describes the order of a set of bifurcations, with respect to the parameter, for *any* family f_μ satisfying the conditions (1)–(4) listed above. The possibility of this fact was indicated to us by numerical studies of Metropolis, Stein, and Stein [5]. For several families of maps, they computed $n(\mu)$ for $n \leqq 15$. For each family they obtained the same sequence, suggesting the possibility of results of the sort we obtain.

2. Local Bifurcation Theory

This section is a review of those parts of dynamical systems theory which are background for the remainder of the paper. One of the basic concepts of dynamical systems is that of *topological equivalence*: if $f, g\colon X \to X$ are two maps of a metric space to itself, then a topological equivalence from f to g is a homeomorphism $h\colon X \to X$ such that $h \circ f = g \circ h$. A topological equivalence maps the f-iterates of a point x to the g-iterates of $h(x)$. If f_μ is a family of maps, then the *regular* values of μ are those which have the property that f_v is topologically equivalent to f_μ for v sufficiently close to μ. If μ is not a regular value, then it is a *bifurcation* value. It is clear from the definition that the set of bifurcation values is a closed set.

We shall be concerned with those bifurcation values which involve periodic orbits. Assume $f_\mu^n(p)=p$ and that n is the smallest integer for which this is true. One says then that p is periodic with *prime period* n. The number $\frac{d}{dx}(f_\mu^n)(p)=\lambda(p)$ can tell one a great deal about the behavior of orbits near p. If $|\lambda|<1$, then f^n is a contraction in some neighborhood of p. Hence, for x close enough to p, $f^{ni}(x)\to p$ as $i\to\infty$. On the other hand, if $|\lambda|>1$ there is some neighborhood U of p such that $f^{ni}(x)\in \mathrm{U}$ for all $i\geqq 0$ implies $x=p$. Moreover, the implicit function theorem implies that when $\lambda\neq 1$ there is a periodic point $p(\mu)$ of prime period n depending smoothly on μ.

Thus the bifurcations of a periodic orbit are of two sorts. The number of periodic orbits of a given prime period n can only change at a value of μ for which there is a periodic point p of period n with $\lambda(p)=1$. The stability of a periodic orbit only changes when $|\lambda|=1$. Bifurcation theory [7] describes the qualitative changes which do take place "generically" in the two cases $\lambda=\pm 1$. We consider these in turn.

Proposition. *Suppose that $f: I\times I\to I$ is a family of maps and that there are p, n, and ν satisfying*

(1) $f_\nu^n(p)=p$,

(2) $\frac{d}{dx}(f_\nu^n)(p)=1$,

(3) $\frac{d^2}{dx^2}(f_\nu^n)(p)>0$,

(4) $\frac{d}{d_\mu}(f_\nu^n(p))>0$.

Then there are intervals (ν_1,ν) and (ν,ν_2) and $\varepsilon>0$ with the properties (1) if $\mu\in(\nu_1,\nu)$, then f_μ^n has no fixed points in $(p-\varepsilon, p+\varepsilon)$, and (2) if $\mu\in(\nu,\nu_2)$ then f_μ^n has two fixed points in $(p-\varepsilon, p+\varepsilon)$, one attracting and one repelling.

Proof. Set $g(x,\mu)=f_\mu^n(x)-x$. The zero set of g is the fixed point set of f^n. Compute that $dg=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial\mu}d\mu$. At the point (p,ν), $\frac{\partial g}{\partial\mu}\neq 0$ and $\frac{\partial g}{\partial x}=0$. Therefore, the implicit function theorem implies that there is a smooth function $\mu=h(x)$ such that $\nu=h(p)$ and $g(x,\mu)=0$ if and only if $\mu=h(x)$ (restricting attention to some neighborhood of $(p,\nu)\in I\times I$). Since $\left.\frac{\partial g}{\partial x}\right|_{(p,\nu)}=0$, $\frac{dh}{dx}(p)=0$. Differentiating $g(x,h(x))=0$ twice, we find $\frac{\partial^2 g}{\partial x^2}+\frac{\partial g}{\partial\mu}\frac{d^2h}{dx^2}=0$. Evaluated at p, $\frac{d^2h}{dx^2}=-\left(\frac{\partial g}{\partial\mu}\right)^{-1}\frac{\partial^2 g}{\partial x^2}>0$. This implies that the curve $\mu=h(x)$ lies to one side of its tangent line at p. The final assertion of the theorem follows upon noting that $\left.\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)\right|_{p,\nu}\neq 0$, so that $\frac{d}{dx}(f_{h(x)}^n(x))$ is a monotonic function at p.

Remark. Changing an inequality in hypotheses (3) or (4) of the proposition changes the sign of $\frac{d^2 h}{dx^2}$ and hence reverses the roles of the intervals (v_1, v) and (v, v_2).

Proposition. Suppose $f: I \times I \to I$ is a family and that there p, n, and v which satisfy

$$f_v^n(p) = p, \tag{1}$$

$$\frac{d}{dx}(f_v^n)(p) = -1, \tag{2}$$

$$\frac{d^3}{dx^3}(f_v^{2n}(p)) < 0, \tag{3}$$

$$\frac{d^2}{dx\,d\mu}(f_\mu^n(x)) > 0 \quad \text{at} \quad (p, v). \tag{4}$$

Then there are intervals (v_1, v) and (v, v_2) and $\varepsilon > 0$ such that

(1) if $\mu \in (v_1, v)$, f_μ^{2n} has exactly one fixed point in $(p-\varepsilon, p+\varepsilon)$. This fixed point is attracting.

(2) if $\mu \in (v, v_2)$, then f_μ^{2n} has three fixed points in $(p-\varepsilon, p+\varepsilon)$.

The largest and smallest form an attracting periodic orbit of period 2 for f_μ^n, and the middle point is repelling.

The proof of this proposition is only slightly more complicated than the proof of the preceding proposition. We remark that $\frac{d}{dx}(f_v^n)(p) = -1$ implies that $\frac{d^2}{dx^2}(f_v^{2n})(p) = 0$. Set $g(x, \mu) = f_\mu^{2n}(x) - x$. Then $g(p) = 0$, $\frac{\partial g}{\partial x}(p) = 0$, $\frac{\partial^2 g}{\partial x^2}(p) = 0$, and $\frac{\partial^3 g}{\partial x^3}(p) \neq 0$. The implicit function theorem implies that there is a smooth function $x = k(\mu)$ such that $p = k(v)$ and $f_\mu^n(k(\mu)) = k(\mu)$. Therefore, $g(k(\mu), \mu) = 0$. Consequently, the function $\bar{g}(x, \mu)$ defined to be $\frac{g(x, \mu)}{x - k(\mu)}$ is a smooth function satisfying $\bar{g}(p, v) = 0$, $\frac{\partial \bar{g}}{\partial x}(p, v) = 0$, $\frac{\partial^2 \bar{g}}{\partial x^2}(p, v) \neq 0$, and $\frac{\partial \bar{g}}{\partial \mu}(p, v) \neq 0$. The arguments of the preceding proposition applied to $\bar{g}$ easily yield the conclusions of this proposition.

3. Symbolic Dynamics

Consider a map $f = f_1: I \to I$ such that $f(0) = f(1) = 0$ and f has a single critical point c. We assume further that $f(c) = 1$. With these assumptions, every point of the interval $[0, 1]$ has exactly two preimages under the map. Since

$$f^3(c) = f^2(c) = f(1) = 0,$$

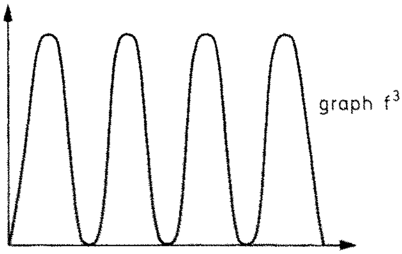

Fig. 2

c has 2^n preimages under the map f^n. The 2^n preimages of c for f^n are critical points of f^{n+1}, and f^{n+1} takes the value 1 at each of these. Between each pair of these preimages of c, there is a critical point of f^n at which f^{n+1} takes the value 0. See Figure 2.

Inside each interval joining a pair of critical points of f^n, f^n is a monotone function on the interval. Therefore, f^n has at least 2^n fixed points, one in each interval joining a pair of critical points. Note that there is no duplication in this count since 0 is the only endpoint which is fixed by any iterate of f. These fixed points are all of the periodic points whose prime period divides n.

Let us partition the fixed points of f^n into sets lying between the same pair of adjacent critical points. We want to describe two different ways of labelling these sets. One way is related to the dynamics of the periodic points, and the other fixes their order on the interval. Let p be a fixed point for the sequence f^n. Associate to p the binary sequence $a_1\, a_2 \ldots a_n$ of length n with the property that p lies in the $\left(\sum_{i=1}^{n} a_i 2^{n-i}\right)$th interval on which f^n is monotonic. Each set of fixed points lying between adjacent critical points is assigned to a different sequence, and each sequence is assigned to a different set of fixed points of f^n. Note that

$$a_i = \begin{cases} 0 & \text{if } \dfrac{df^{i-1}}{dx}(p) > 0 \\ 1 & \text{if } \dfrac{df^{i-1}}{dx}(p) < 0. \end{cases}$$

On the other hand, we can associate to p a binary sequence $b_1 \ldots b_n$ of length n by the rule $b_i = 0$ if $f^{i-1}(p)$ lies to the left of c, and $b_i = 1$ if $f^{i-1}(p)$ lies to the right of c. Analytically, $b_i = \frac{1}{2}\left(-1 + \operatorname{sgn} \frac{df}{dx}(f^{i-1}(p))\right)$. Observe that $a_1 = b_1$. Since $\frac{df^i}{dx}(p) = \frac{df}{dx}(p)\frac{df}{dx}(f(p)) \cdots \frac{df}{dx}(f^{i-1}(p))$, we find that $a_i = b_1 + \cdots + b_i \pmod 2$. From this it follows that $a_i + a_{i-1} = b_i \pmod 2$ for $i > 1$.

Still assuming that $f^n(p) = p$, we note that the sequence of b_i's associated to the point $f(p)$ is the sequence $b_2\, b_3 \ldots b_n\, b_1$. Therefore, cyclic permutation of indices of the b sequences allows us to construct a coarser partition of the fixed points of

a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1
0	0	1	0	0	0	1	1
0	0	1	1	0	0	1	0
0	1	0	0	0	1	1	0
0	1	0	1	0	1	1	1
0	1	1	0	0	1	0	1
0	1	1	1	0	1	0	0
1	0	0	0	1	1	0	0
1	0	0	1	1	1	0	1
1	0	1	0	1	1	1	1
1	0	1	1	1	1	1	0
1	1	0	0	1	0	1	0
1	1	0	1	1	0	1	1
1	1	1	0	1	0	0	1
1	1	1	1	1	0	0	0

Sequences joined by solid lines represent a single orbit.

Fig. 3. Partitioning the fixed points of f^4

f^n. This coarser partition lumps together all fixed points of f^n whose f orbits contain a point lying in a specific interval between two adjacent critical points of f^n. Converting the b_i sequences into a_i sequences tells us the relative order of points on the unit interval. In Figure 3, we illustrate the partitions for $n=4$.

This procedure allows us to partition the fixed points of f^n with orbits lying in different subintervals between critical points into f-orbits and determine their relative order on the interval. To compare the order in I of the points in periodic orbits of periods n_1 and n_2, we perform the above construction with $n=n_1 \cdot n_2$. This allows us to compare the order in I of the points belonging to periodic orbits of periods n_1 and n_2 since $f^{n_1 \cdot n_2}$ fixes all such points. In this way, we can partition the set of periodic points of f of all orders and determine the order of their points relative to each other, except for those periodic points which lie in the same subintervals joining adjacent critical points of an iterate of f.

4. The Global Bifurcation Diagram

Let $f: I \times I \to I$ be a family of maps satisfying properties (1)–(4) of the Introduction. Assume further that f is "generic" in the sense that (1) the hypotheses of the propositions of section II are satisfied at all points for which $f_\mu^n(x)=x$ and $\left|\frac{d}{dx} f_\mu^n(x)\right|=1$ and (2) there are no points satisfying these two equations when $\mu=1$. Denote by P_n the set of solutions of the equation $f_\mu^n(x)=x$. The genericity assumption implies that P_n is a union of non-singular smooth curves. The properties (1)–(4) imply that the intersection of P_n with the interval $\mu=0$ is the origin, while

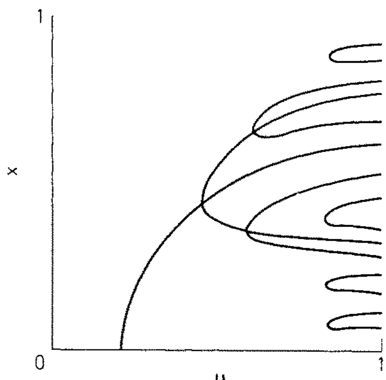

Fig. 4. The set P_4

the intersection of P_n with the line $\mu=1$ consists of at least 2^n distinct points. The local bifurcation theory allows us to assign a stability index (s for stable, u for unstable) to almost all points of P_n according to whether the absolute value of $\frac{d}{dx} f_\mu^n(p)=\lambda(p)$ is smaller or larger than 1. The exceptional points occur as either vertical tangencies of $P_n(\lambda(p)=1)$ or at points for which the stability index changes and $\overline{P_{2n}-P_n}$ contains a point of intersection with $P_n(\lambda(p)=-1)$. Figure 4 illustrates the simplest possible diagram for P_4.

Write P_n as a union $\cup\gamma_i^n$ of maximal smooth, connected, nonsingular curves γ_i^n in P_n. Since f_1 has no periodic orbits for which $\lambda(p)=1$, the number of points of $P_n\cap\{\mu=1\}$ inside each subinterval joining adjacent critical points of f_1^n is odd. In each subinterval on which f_1^n is decreasing, there must be exactly one point p for which $f_1^n(p)=p$, and at this point $\lambda(p)<-1$. We want to focus our attention on those curves γ_i^n which have endpoints p for which $\lambda(p)<-1$. To state the main result about these points, we need to establish some additional notation. If $(p,1)\in P_n$, denote $m(p)$ to be the minimum value of μ on the curve γ_i^n having an endpoint at p. Clearly, $m(p)$ is constant on the points in one f-orbit. Denote by $\Gamma(p)$ the union of those γ_i^n passing through points in the f_1-orbit of p and by $M(p)$, the maximum value of x in the f_1 orbit of p. A key idea of this paper is contained in the following theorem.

Theorem. *Let f: $I\times I\to I$ be a family of maps satisfying properties* (1)–(4) *of the Introduction and the genericity conditions of the beginning of this section.*

Let p_1 and p_2 be two fixed points of f_1^n such that $\frac{d}{dx}(f_1^n(p_i))<0$ for $i=1,2$ and such that p_1 and p_2 lie in different f_1 orbits. Then $M(p_1)<M(p_2)$ if and only if $m(p_1)<m(p_2)$.

In other words, the largest values of orbits on the line $\mu=1$ determines the order in which these periodic orbits arose through bifurcation—larger values indicating later bifurcations.

Proof. Consider the function $\lambda(x,\mu)=\frac{d}{dx} f_\mu^n(x)$ on P_n and a curve γ_i^n intersecting

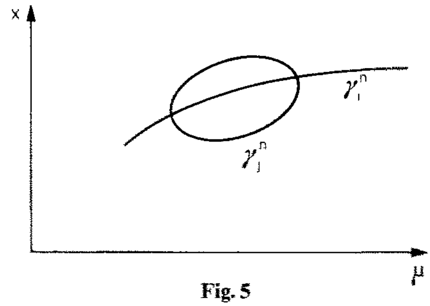

Fig. 5

the line $\mu=1$ at a point p for which $\lambda(p,1)<0$. At a point $(q,v)\in\gamma_i^n$ with $v=m(p)$, we know from the local bifurcation theory that γ_i^n has a vertical tangent and that $\lambda(q,v)=1$. Since $\lambda(p,1)<0$, there is a point (r,ρ) of γ_i^n at which $\lambda(r,\rho)=0$. The orbit of r under f_ρ contains the critical point $c(\rho)$ since $\lambda(r,\rho)=0$. Therefore, the orbit of r also contains the maximum value $f_\rho(c)$ of $f_\rho(x)$. If p' is another fixed point of f_1^n for which $\lambda(p',1)<0$ and if $m(p')<m(p)$, then every point of $\Gamma(p')$ intersected with the line $\mu=\rho$ must lie below the point $f_\rho(c)$.

To use this fact, we examine the intersections of γ_i^n and $\Gamma(p')$. The only points of intersection of γ_i^n and $\Gamma(p')$ occur at points of the kind described in the second proposition of Section II. Let p have prime period k. If $(x,\sigma)\in\Gamma(p')\cap\gamma_i^n$, then the component γ_j^n of $\Gamma(p')$ containing (x,σ) represents a periodic orbit of period 2 for f^k. If γ_j^n has another intersection with γ_i^n, then γ_j^n must be a simple closed curve as shown in Figure 5.

The branches of γ_j^n above and below γ_i^n must meet at each point of intersection with γ_i^n since they represent one orbit of f^k of period 2. Since $\Gamma(p')$ intersects the line $\mu=1$, we conclude that γ_j^n can intersect γ_i^n in at most one point (x,σ).

If there are no points of intersection of $\Gamma(p')$ and γ_i^n, then $M(p')<M(p)$ since $\Gamma(p')\cap\{\mu=\rho\}$ lies below $f_\rho(c)$. If there is a point $(x,\sigma)\in\Gamma(p')\cap\gamma_i^n$, then $\sigma>\mu$ and $M(p')>M(p)$. Moreover, the observation that $\Gamma(p')\cap\{\mu=\rho\}$ lies below $f_\rho(c)$ implies that $m(p')>m(p)$. We conclude that $m(p')<m(p)$ implies $M(p')<M(p)$. Note that $m(p')<m(p)$ implies that $M(p')\neq M(p)$ since p and p' lie in different f-orbits.

The theorem will be proved if we eliminate the possibility that $m(p)=m(p')$ and $M(p)\neq M(p')$ with $\lambda(p)$ and $\lambda(p')$ both negative. If this happens, let γ_i^n, γ_j^n be curves of P_n containing $M(p)$, $M(p')$ and let (r,ρ), (r',ρ') be points on γ_i^n, γ_j^n at which $\lambda(r,\rho)=\lambda(r',\rho')=0$. Now r is larger than any point in the f-orbit of $\gamma_j^n\cap\{\mu=\rho\}$, and r' is larger than any point in the f-orbit of $\gamma_i^n\cap\{\mu=\rho'\}$. Since $m(p)=m(p')$, both intersections are non-empty. The curves γ_i^n and γ_j^n must cross. We have already observed, however, that if γ_i^n and γ_j^n intersect, then $m(p)\neq m(p')$. Hence the theorem is proved: if p_1 and p_2 satisfy the hypotheses of the theorem, then the order of $M(p_1)$ and $M(p_2)$ and the order of $m(p_1)$ and $m(p_2)$ must be the same.

Remark 1. Combining the theorem above with the symbolic dynamics allows us to order a whole set of bifurcation values giving "birth" to those fixed points p of f_1 for which $\lambda(p,1)<0$. The sequence of bifurcations is the same as that determined by choosing the largest members of the f_1 orbits of these fixed points. This last sequence is determined by the symbolic dynamics of the map f_1 on its periodic orbits and is independent of the particular generic family of maps satisfying properties (1)–(4); hence the bifurcation sequence we have determined is independent of the map.

Remark 2. We want to eliminate the genericity hypotheses from the theorem. To do so, we have to define $m(p)$ for a family f which need not be generic. If $(p, 1)$ is a periodic point of f, set $m(p)$ to be the lim inf of $m(q)$ for generic families which are perturbations of f having a periodic point $(q, 1)$ near $(p, 1)$. With this definition of $m(p)$ for nongeneric families, the theorem is still true. Assume $(p, 1)$, $(p', 1)$ are periodic points of f with $\lambda(p, 1)<0$, $\lambda(p', 1)<0$ and $M(p)<M(p')$. We can find a generic perturbation g of f with periodic points $(q, 1)$, $(q', 1)$ near $(p, 1)$, $(p', 1)$ such that $M(q)<M(q')$ and $|m(q)-m(p)|<|m(p)-m(p')|$. Since $m(q)<m(q')$, we conclude that $m(p)\leqq m(p')$. The same argument that was used in the theorem implies that $m(p)\neq m(p')$.

Remark 3. We have proved more than the theorem states. Inside γ_i^n, on the portion joining $(p_1, 1)$ to its intersection with the line $\mu=m(p_1)$, all of the vertical tangencies occur over the interval $(m(p_1), m(p_2))$. Indeed, we must have $\rho_1<m(p_2)$, so the derivative of f_n is negative on the component of $(p_1, 1)$ in $\gamma_i^n\cap\{m(p_2)\leqq\mu\leqq 1\}$.

Remark 4. The theorem as stated applies to the fixed points p of f_1^n with $\lambda(p)<0$. There are 2^{n-1} such points. In a generic family and for odd n, each such point is connected by a curve γ_i^n to another periodic point with $\lambda>0$. (For even n, there are bifurcations of orbits of period $n/2$ with $\lambda=-1$ which yield a more complicated structure for P_n. See Fig. 4.) Since the curves γ_i^n do not cross, the order of $M(p)$ vs. the order of $m(p)$ is preserved for these points as well. Thus the theorem applies to all those periodic points which lie in curves γ_i^n having an endpoint p with $\lambda(p)<0$. If p_1 and p_2 are the endpoints of γ_i^n and if $\lambda(p_1)<0$, then $M(p_1)>M(p_2)$. Thus there are 2^n periodic points of f_n which can be partitioned into orbits for which one can determine the sequence of bifurcations giving rise to these orbits. By considering bifurcations with $\lambda=-1$, the same can be done for even n as well as odd n.

We end this section by summarizing some of the remarks in a theorem. Consider the map $g\colon I\to I$ defined by

$$g(x)=\begin{cases}2x & 0\leqq x\leqq\frac{1}{2}\\ 2(1-x) & \frac{1}{2}\leqq x\leqq 1.\end{cases}$$

Order the set P of periodic orbits of g by the largest points in their orbits: the orbit π_1 is larger than the orbit π_2 if π_1 contains a point larger than all of the points of π_2; i.e., $\max_{\pi_1} x>\max_{\pi_2} x$. Define the map $\gamma\colon P\to\mathbb{Z}$ which assigns to an element of P its period.

Theorem. *Let $f_\mu\colon I\times I\to I$ be a family of maps with the following properties*

(1) *$f_\mu(x)$ is a smooth function of both x and μ (at least C^3).*
(2) *$f_\mu(0)=f_\mu(1)=0$ for all $\mu\in I$.*
(3) *f_μ has a single critical point $c_\mu\in I$ for each $\mu>0$.*
(4) *f_0 is identically zero, and $f_1(c_1)=1$.*

Denote by $B \subset I$ the set of bifurcation values for periodic orbits of f_μ and $\beta\colon B \to \mathbb{Z}$ the map which assigns the period of the bifurcating periodic orbit to a bifurcation value. Then there is an order perserving map $\alpha\colon P \to B$ such that the diagram

$$\begin{tikzcd} & \mathbb{Z} & \\ P \arrow[ur, "\gamma"] \arrow[rr, "\alpha"'] & & B \arrow[ul, "\beta"'] \end{tikzcd}$$

commutes.

5. Period $2n+1$ Implies Period $2n-1$

Let f be a smooth map of the interval with a single critical point and $f(0)=f(1)=0$. From the results of Section 4, it follows that there is an ordering of the positive integers (independent of f) such that if n_1 precedes n_2 and f has a periodic orbit of period n_1, then f also has a periodic orbit of period n_2. This ordering has been determined by Sharkovsky [6]. (His result is valid for all continuous maps of the interval.) We deduce the ordering here by a computation based upon the symbolic dynamics of Section 3. We use the language of Section 3, referring to binary sequences determining order on the interval as a-sequences and to the sequences obtained by examining the derivatives of f on the iterates of a point as b-sequences. This section illustrates the kinds of applications which can be obtained from our preceding results.

Given a positive integer n, we want to determine the b-sequence which yields the smallest value of μ for which there is an orbit of prime period n in a family f_μ^n. Following the results of Section 4, this calculation can be made by examining a particular map g with $g(0)=g(1)=0$ and the single critical value 1. It is convenient to use the map g defined by

$$g(x)=\begin{cases} 2x & 0 \leqq x \leqq \frac{1}{2} \\ 2-2x & \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Though this map is not smooth, it is topologically conjugate to the map $f(x)=4x(1-x)$ [8]. Therefore, the order of its periodic orbits is the same as the order for a smooth map. If p is a periodic point of q of period n with a-sequence $a_1 \dots a_n$, then the binary expansion of p is $a_1 \dots a_n a_1 \dots a_n a_1 \dots a_n \dots$. For each n, we want to find the periodic point p_n of prime period n with the properties that (1) p_n is the largest point in its orbit, and (2) every other periodic orbit of prime period n contains a point which is larger than p_n. We shall call p_n the *min-max of period n.* The results of Sections 3 and 4 imply that the ordering of p_n in I is the ordering of the positive integers we seek. We shall give an algorithm for explicitly computing the b-sequences (and hence the a-sequences and binary expansions) of each p_n. As the proof is a rather intricate combinatorial argument, we begin with a special case.

Proposition. *Let n be an odd integer, and let $b_1 \ldots b_n$ be the b-sequence of the* min-max p_n *of period n. Then $b_2=0$ and $b_i=1$ for $i \neq 2$.*

Proof. The proof proceeds by a series of small observations about the a-sequence of the min-max p_n. First, the b-sequence of p_n contains both 0 and 1 since p_n is not a fixed point. Recall that if the b-sequence of p_n is $b_1 b_2 \ldots b_n$, then the b-sequence of $f(p)$ is $b_2 \ldots b_n b_1$. Note that the orbit of p_n contains a point with b-sequence beginning 10 and (consequently) a-sequence beginning 11. This implies that the periodic point p with the b-sequence $10111 \ldots 1$ is the largest point in its orbit. From this also follows the second observation that the b-sequence of p_n begins 10 (otherwise, there is a larger point in its orbit). The third observation is that the b-sequence of p_n does not have 2 consecutive 0's. If it did, there would be a point in its orbit with b-sequence beginning 100 and a-sequence beginning 111. The a-sequence associated to p is $110\ldots$. Therefore, the orbit of p contains no point as large as one with b-sequence beginning $100\ldots$. So, the b-sequence of p_n does not have consecutive 0's. Assume now that the b-sequence of p_n contains more than one zero. The fourth observation is that a cyclic sequence of 0's and 1's of odd length without consecutive 0's contains a string of 1's of even length. The assumption implies that the orbit of p_n contains a point q with b-sequence beginning $10\underbrace{11\ldots1}_{2k}0\ldots$, where the second 0 occurs as the $(2k+3)rd$ term of the sequence. The a-sequence of q has a 1 in the $(2k+3)\,rd$ term since $a_i=\sum_{j\leq i} b_j$. Therefore, the binary expansion of p agrees with the binary expansion of q in the first $(2k+2)$ places, but q has a 1 in the $(2k+3)\,rd$ place of its expansion compared to a 0 in the $(2k+3)\,rd$ place of the expansion for p. Therefore $q>p$. We conclude that p is the min-max p_n of period n.

Corollary. *Let $f: I \to I$ be a map with $f(0)=f(1)=0$ and a single critical point. If f has a periodic orbit of odd period, then f has periodic orbits of all longer odd periods.*

Proof. Compare the binary expansions of the min-max's p_n of odd period n. These are calculated to be $\frac{2^n}{2^n-1}\left(\frac{1}{2}+\frac{2^{n-1}-1}{3\cdot 2^{n-1}}\right)$ from the b-sequences of p_n. These numbers decrease with n. Therefore, if f has a periodic orbit of odd period n, it has periodic orbits of all longer odd periods.

We now proceed to extend the computations to find the b-sequences of min-max's of even periods. From this we shall obtain the ordering of Sharkovsky. We begin by examining the min-max's of period 2^l.

Proposition. *There is an algorithm for determining the b-sequence of the* min-max *P_{2^l} of period 2^l. If S_l denotes the b-sequence of P_{2^l}, then $S_{l+1}=S_l\,S_{l-1}\,S_{l-1}$.*

Proof. First, note that if $p_n<p_{2^l}$, then 2^l is divisible by n. This follows from the results of previous sections since each stable periodic orbit of a family becomes unstable at a bifurcation with $\lambda=-1$. This bifurcation gives rise to a stable periodic orbit of twice the period. Therefore, the first sequence of bifurcations in a family yields stable periodic orbits whose periods are a power of 2. We conclude that $p_1<p_2<p_4<\cdots$ and that if n is not a power of 2, $p_{2^l}<p_n$.

Next, explicitly compute that $S_1=1$ and $S_2=10$. Assume that we have determined the sequences $S_1, \ldots, S_l$, and that they satisfy $S_{k+1}=S_k S_{k-1} S_{k-1}$ for $k<l$. We assert that the sequence S_{l+1} is given by $S_l S_{l-1} S_{l-1}$. Denote by S_l' the sequence obtained by changing the last term of S_l. The assertion that $S_{l+1}=S_l S_{l-1} S_{l-1}$ is the same as the assertion that $S_{l+1}=S_l S_l'$ or $S_{l+1}'=S_l S_l$. Consider the b-sequence $b_1 \ldots b_{2^{l+1}}=S_l S_l'$ associated to a periodic point p. We need to prove that p is the largest point in an orbit of prime period 2^{l+1}. That p is a point of prime period 2^{l+1} is evident since $b_{2^l} \neq b_{2^{l+1}}$. Now consider another point q in the orbit of p. Writing $S_l S_l'=S_{l-1} S_{l-1}' S_{l-1} S_{l-1}$ by the induction hypothesis, the first 2^l terms of the b-sequence of q agree with the b-sequence of a point of the orbit of p_{2^l} which is not p_{2^l} or with a cyclic permutation of the sequence $S_{l-1} S_{l-1}$. In either case, by comparing the a-sequences of p_{2^l} and q and remembering that $p_{2^{l-1}}<p_{2^l}$, we conclude that $q<p$. Therefore, p is the largest point of its orbit. We now need to prove that p is the smallest of the largest points in orbits of prime period 2^{l+1}. We know that $p_{2^{l+1}}>p_{2^l}$. By comparing the binary expansions of $p_{2^{l+1}}$ and p_{2^l}, we note that they must differ in the first 2^{l+1} positions since both are repeating with period 2^{l+1}. Thus the a-sequence of $p_{2^{l+1}}$ must represent a binary number which is larger than the binary number represented by the a-sequence associated to the b-sequence $S_l S_l$. (These are the a-sequence and b-sequence of p_{2^l} regarded as a periodic point of period 2^{l+1}.) The a-sequence associated to the b-sequence $S_l S_l$ ends with 0 (recall $a_i=\sum_{j=i} b_j$,) so the a-sequence associated to $S_l S_l'$ is a lower bound for the a-sequence of $p_{2^{l+1}}$. Therefore $p=p_{2^{l+1}}$, proving the proposition. Note that the last term of the a-sequence of S_l is a 1 since S_l contains an odd number of 1's. This fact is used implicitly in later computations.

We can now describe explicitly the b-sequence of the min-max of period n in terms of the sequences S_l.

Proposition. *Let $n=2^l \cdot m$ be a positive integer with $m>1$ odd. Denote by S_l the b-sequence of the* min-max *p_{2^l} of period 2^l. Then the* min-max *p_n of period n is the sequence $S_{l+1} S_l \ldots S_l$ consisting of the concatenation of S_{l+1} with $(m-2)$ blocks of the form S_l.*

Remark. Continuing to denote S_l' the sequence obtained by changing the last term of S_l, the b-sequence of p_n can be written $S_l S_l' S_l \ldots S_l$. This expression generalizes the one for p_n when n is odd.

Proof. First, note that the sequence $S_l S_l' S_l \ldots S_l$ represents a point p of prime period $n=2^l \cdot m$. Next we prove that p is the largest point of its orbit. Let $q=g^i(p)$, $1 \leqq i \leqq n$. If $i>2^{l+1}$, then the first 2^{l+1} terms of the b-sequence of q are a cyclic permutation of the sequence $S_l S_l$. Since $p_{2^l}<p_{2^{l+1}}$ and $S_l S_l'=S_{l+1}$, this implies that $q<p$. If $i \leqq 2^{l+1}$, then the first 2^{l+1} terms of the b-sequence of q are a non-trivial permutation of S_{l+1}. Once again comparing binary sequences of length 2^{l+1} reveals that $q<p$. So p is the largest point in its orbit.

The last part of the proposition is a proof that any periodic orbit of prime period n contains a point as large as n. This is done in two steps. Denote the b-sequence of p_n by $b_1 \ldots b_n$. There is a largest k with the property that $b_1 \ldots b_n$ is the concatenation of sequences of the form S_k and S_k'. (If this is true for one k, it is true for smaller k since $S_k = S_{k-1} S'_{k-1}$ and $S'_k = S_{k-1} S_{k-1}$ for $k > 1$.) The first of the two steps is that $k = l$. The final step will be that the block S'_l occurs only once in the b-sequence of p_n.

Consider $b_1 \dots b_n$ as the concatenation $B_1 \dots B_r$, where $r = 2^{l-k} \cdot m$ and $B_i = S_k$ or S'_k. If $k < l$, we shall prove that $b_1 \dots b_n$ can be written as a concatenation of blocks of the form S_{k+1} and S'_{k+1}. Observe that both S_k and S'_k occur in the sequence $B_1 \dots B_r$ since p_n has prime period n. This implies $B_1 = S_k$ and $B_2 = S'_k$ since the a-sequence associated to S_k is larger than the a-sequence associated to S'_k, and the a-sequence associated to $S_k\, S'_k$ is larger than the a-sequence associated to $S_k\, S_k$. Continuing, if $B_i = S'_k$, then $B_{i+1} = S_k$ since the a-sequence associated to $S_k\, S'_k\, S'_k$ is larger than that associated to $S_k\, S'_k\, S_k$. Similarly, the a-sequence of $S_k\, S'_k\, S_k\, S_k\, S'_k$ is larger than that of $S_k\, S'_k\, S_k\, S_k\, S_k$ and the a-sequence of $S_k\, S'_k\, S_k\, S_k\, S_k\, S_k$ is larger than that of $S_k\, S'_k\, S_k\, S_k\, S_k\, S'_k$.

Assume now that $k < l$ and consider the sequence $S_{k+1}\, S'_{k+1}\, S_{k+1} \dots S_{k+1}$. Written as a concatenation of blocks of the form S_k and S'_k this sequence has the following properties: (1) it has prime period n, (2) it begins $S_k\, S'_k$, (3) S'_k occurs only in blocks with even index. The comparisons made above show that the sequences for p_n must also have these properties. A sequence satisfying the third property can be written as a concatenation of blocks of the form S_{k+1} and S'_{k+1} since $S_{k+1} = S_k\, S'_k$ and $S'_{k+1} = S_k\, S_k$. This contradicts the assumption on k, so we conclude that $k = l$.

The final step of the proof proceeds much as the determination of the b-sequence of p_n when n is odd. We have proved that the b-sequence of p_n is a concatenation of $B_1 \dots B_m$ of m blocks of the form S_l or S'_l. If two blocks of the form S'_l occur, then the orbit of p_n contains a point whose b-sequence begins with the blocks $S_l\, S'_l \underbrace{S_l\, S_l \dots S_l}_{2k} S'_l$. The first two blocks are $S_l\, S'_l$ and the next occurrence is in a block with odd index $(2k+3)$. The point whose b-sequence begins this way is larger than any point in the orbit of the point p with b-sequence $S_l\, S'_l\, S_l \dots S_l$ of length n. We conclude that $p = p_n$, proving the proposition.

Finally the result of Sharkovsky is recovered by comparing the binary expansions of the p_n obtained from their b-sequences. Consider $m = 2^k \cdot r$ and $n = 2^l \cdot s$ with r, s odd and larger than 1. If $k > l$, the b-sequence of p_m begins $S_l\, S'_l\, S_l\, S_l\, S_l\, S'_l$. The b-sequence of p_n is $S_l\, S'_l\, S_l$ (if $s = 3$), or $S_l\, S'_l\, S_l\, S_l\, S_l$ (if $s = 5$), or it begins $S_l\, S'_l\, S_l\, S_l\, S_l\, S_l$. In each of these three cases, a comparison of binary expansions yields $p_m < p_n$. If $k = l$ and $r > s$ then the same sort of computation we used for odd periods shows $p_m < p_n$. The point p_n, regarded as a periodic point of period $2n$ has a b-sequence which begins $S_l\, S'_l \underbrace{S_l \dots S_l}_{(s-1)} S'_l$ while the b-sequence of period p_m begins $S_l\, S'_l \underbrace{S_l \dots S_l}_{s} S_l$. Comparison of the corresponding a-sequences shows that $p_m < p_n$. Thus we have

Theorem (Sharkovsky). Let T be the ordered set $\{3 < 5 < 7 \cdots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < \cdots < 4 \cdot 3 < 4 \cdot 5 < 4 \cdot 7 < \cdots < 8 < 4 < 2 < 1\}$. Let $f\colon I \to I$ be a smooth map such that $f(0) = f(1) = 0$ and f has a single critical point. If $m < n$ relative to the order of T and f has a periodic point of prime period m, then f has a periodic point of period n.

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