FACTORING GLEASON POLYNOMIALS MODULO 2

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Abstract. Among the connected components of the interior of the Mandelbrot set are those that are *hyperbolic*. These components consist of parameters $c \in \mathbb{C}$ for which the critical point $z_0 = 0$ of $f_c : z \mapsto z^2 + c$ is attracted to an attracting periodic cycle. Every hyperbolic component contains a unique *center*; that is, a parameter c for which the critical point z_0 is periodic. For a given $n \geq 1$, the *Gleason polynomial for period* n is the monic polynomial $G_n \in \mathbb{Z}[c]$ whose roots are exactly the centers of the hyperbolic components of period n. It is unknown if G_n factors over $\mathbb{Z}$. In this article, we factor G_n modulo 2. We prove the following remarkable fact: the number of irreducible factors of G_n modulo 2 is equal to the number of real roots that G_n has in $\mathbb{C}$.

1. Introduction

Let $\mathbf{k} = \mathbb{Q}$ be the field of rational numbers or $\mathbf{k} = \mathbb{F}_2$ be the finite field with 2 elements. Let $\overline{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$.

Given $c \in \overline{\mathbf{k}}$, denote by $f_c \in \mathbf{k}[z]$ the quadratic polynomial

$$f_c(z) := z^2 + c.$$

A point $z \in \overline{\mathbf{k}}$ is a *periodic point* for f_c if $f_c^{\circ n}(z) = z$ for some positive integer n. The least such integer is called the *period* of z.

If z is periodic of period $n \geq 1$ for f_c, then (c, z) is a zero of the polynomial $F_n \in \mathbf{k}[c, z]$ defined by

$$F_n(c, z) := f_c^{\circ n}(z) - z.$$

An elementary induction shows that F_n has integer coefficients, degree 2^{n-1} with respect to c and degree 2^n with respect to z. The coefficient of $c^{2^{n-1}}$ is 1 and the coefficient of z^{2^n} is 1. If m divides n, then $F_m(c, z) = 0 \Rightarrow F_n(c, z) = 0$. In addition, the polynomial

$$F_m(0, z) = z^{2^m} - z \in \mathbf{k}[z]$$

has simple roots, so that F_m is square-free. Thus, if m divides n, then F_m divides F_n in $\mathbf{k}[c, z]$. It follows that there exists a sequence $(\Phi_n \in \mathbf{k}[c, z])_{n \geq 1}$ such that

$$F_n(c, z) = \prod_{m|n} \Phi_m(c, z).$$

The polynomial Φ_n is called the n-th *dynatomic* polynomial.

Let $\mu : \mathbb{N} \setminus \{0\} \to \{-1, 0, 1\}$ be the Möbius function, and let $(\delta_n)_{n \geq 1}$ be the sequence of integers defined by

$$\delta_n := \sum_{m|n} \mu(m) 2^{\frac{n}{m}}.$$

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Then, the polynomial Φ_n has degree $\delta_n/2$ with respect to c and degree δ_n with respect to z. In this article, we are interested in the factorization of Φ_n in $\mathbf{k}[c,z]$.

The following result is due to Bousch [B].

Theorem 1.1. *If $\mathbf{k}=\mathbb{Q}$, then for $n\geq 1$, the dynatomic polynomial Φ_n is irreducible in $\mathbb{Q}[c,z]$.*

We shall see that when $\mathbf{k}=\mathbb{F}_2$, the situation is radically different. Let $(\gamma_n)_{n\geq 1}$ be the sequence defined by

$$\gamma_n := \frac{1}{2n}\sum_{\substack{m\mid n\\ m \text{ is odd}}} \mu(m)2^{\frac{n}{m}}.$$

Theorem 1.2. *If $\mathbf{k}=\mathbb{F}_2$, then for $n\geq 1$, the dynatomic polynomial Φ_n has exactly γ_n irreducible factors in $\mathbb{F}_2[c,z]$ which are monic with respect to c. These are of the form $Q(z^2+c-z)$ with $Q\in\mathbb{F}_2[c]$. If n is odd, then each factor has degree $2n$ with respect to z and degree n with respect to c. If n is even, then there are $\gamma_{n/2}$ factors of degree n with respect to z and degree $n/2$ with respect to c, and there are $\gamma_n-\gamma_{n/2}$ factors of degree $2n$ with respect to z and degree n with respect to c.*

Our proof in §6 relies on studying the restriction of Φ_n to the slice $\{z=0\}$. Note that 0 is a critical point of f_c, i.e., $f_c'(0)=0$.

Remark 1.3. For $\mathbf{k}=\mathbb{F}_2$, all points are critical points since the derivative of f_c identically vanishes.

A parameter $c\in\overline{\mathbf{k}}$ is called a *center of period n* if 0 is a periodic point of period n for f_c. The centers of period n are the roots of the polynomial $G_n\in\mathbf{k}[c]$ defined by

$$G_n(c)=\Phi_n(c,0).$$

The polynomial G_n is called the n-th *Gleason polynomial*. It has degree $\delta_n/2$.

The factorization of Gleason and related polynomials in $\mathbb{Q}[c]$ as well as in $\mathbb{F}_2[c]$ has recently attracted attention (see for example [G1] and [G2]); in particular, there are implications regarding the irreducibility over $\mathbb{C}$ of some dynamically defined curves in the space of quadratic rational maps (see for example [BEK]). Here is a long-standing conjecture whose origin is unknown to us. See Remark 3.5 in [M].

Conjecture 1.4. *If $\mathbf{k}=\mathbb{Q}$, then for $n\geq 1$, the polynomial G_n is irreducible in $\mathbb{Q}[c]$.*

In the following statement, $\mathcal{M}$ is the Mandelbrot set. For the notion of primitive or satellite hyperbolic components, see §4.4. The following result is due to Lutzky [L1], [L2]. We shall present a proof in §4.4 that differs from Lutzky's proof (see §4.5 for a discussion of Lutzky's proof).

Theorem 1.5. *If $\mathbf{k}=\mathbb{Q}$, then G_n has exactly γ_n roots in $\mathbb{R}$. When n is odd, these γ_n roots are centers of primitive components of $\mathcal{M}$. When n is even, $\gamma_{n/2}$ of these roots are centers of satellite components of $\mathcal{M}$ and $\gamma_n-\gamma_{n/2}$ of these roots are centers of primitive components of $\mathcal{M}$.*

We shall prove that when $\mathbf{k}=\mathbb{F}_2$, there is a parallel count for the number of monic irreducible factors of the n-th Gleason polynomial G_n. A polynomial $P\in\mathbf{k}[c]$ of degree d is *centered* if the coefficient of c^{d-1} is equal to 0. Otherwise it is *noncentered*.

Theorem 1.6. *If $\mathbf{k} = \mathbb{F}_2$, then G_n has exactly γ_n monic irreducible factors in $\mathbb{F}_2[c]$. When n is odd, those factors are the γ_n irreducible centered polynomials of degree n in $\mathbb{F}_2[c]$. When n is even, those factors are the $\gamma_{n/2}$ irreducible noncentered polynomials of degree $n/2$ in $\mathbb{F}_2[c]$ together with the $\gamma_n - \gamma_{n/2}$ irreducible centered polynomials of degree n in $\mathbb{F}_2[c]$.*

This will be proved in §5.

Remark 1.7. Theorem 1.6 generalizes to cases where the degree d is a power of a prime. We state this generalization without proof in Theorem 1.8. We introduce the following notation in order to include the statement of the theorem in this more general case.

Let $d = p^r$, where p is a prime number, and r is a positive integer. Let κ_n and ρ_n be the following sequences

$$\kappa_n = -\frac{(d-1)p}{dn} \sum_{p|m|n} \mu(m) d^{\frac{n}{m}}$$

and

$$\rho_n = \frac{1}{dn} \sum_{m|n} \mu(m) d^{\frac{n}{m}} - \frac{(p-1)(d-1)}{dn} \sum_{p|m|n} \mu(m) d^{\frac{n}{m}}.$$

Theorem 1.8. *If $\mathbf{k} = \mathbb{F}_d$, then G_n has exactly ρ_n monic irreducible factors in $\mathbb{F}_d[c]$. When p does not divide n, those factors are the ρ_n irreducible centered polynomials of degree n in $\mathbb{F}_d[c]$. When p does divide n, those factors are the κ_n irreducible noncentered polynomials of degree n/p in $\mathbb{F}_d[c]$ together with the $\rho_n - \kappa_n$ irreducible centered polynomials of degree n in $\mathbb{F}_d[c]$.*

Remark 1.9. One might hope that there is a corresponding generalization of Theorem 1.5 to multibrot sets associated to prime powers. Unfortunately, the choice of what hyperbolic components to count is not clear. For example, there are no hyperbolic components with real centers if p is odd.

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2. Some sequences of integers

Recall that $\mu : \mathbb{N} \setminus \{0\} \to \{-1, 0, 1\}$ is the Möbius function and that for $n \geq 1$,

$$\delta_n := \sum_{m|n} \mu(m) 2^{\frac{n}{m}},$$

so that

$$2^n = \sum_{m|n} \delta_m.$$

It will be convenient to consider the sequence $(\varepsilon_n)_{n\geq 1}$ defined by

$$\varepsilon_n := -\sum_{\substack{m|n \\ m \text{ is even}}} \mu(m) 2^{\frac{n}{m}} \quad \text{so that} \quad \delta_n + \varepsilon_n = \sum_{\substack{m|n \\ m \text{ is odd}}} \mu(m) 2^{\frac{n}{m}}.$$

Lemma 2.1. *The sequence $(\varepsilon_n)_{n\geq 1}$ is characterized by the recursion*

$$\forall n \geq 1, \quad \varepsilon_{2n-1} = 0 \quad \textit{and} \quad \varepsilon_{2n} = \delta_n + \varepsilon_n. \tag{1}$$

Remark 2.2. This shows that the sequence $(\varepsilon_n)_{n\geq 1}$ takes nonnegative values.

Remark 2.3. This lemma asserts that any sequence satisfying recursion (1) is equal to the sequence $(\varepsilon_n)_{n\geq 1}$.

Proof. First, assume $n \geq 1$. Since $2n-1$ does not have any even divisor, the sum defining ε_{2n-1} is empty, so that $\varepsilon_{2n-1} = 0$. In addition,

$$\begin{aligned}
\delta_n + \varepsilon_n &= \sum_{m|n} \mu(m) 2^{\frac{n}{m}} - \sum_{\substack{m|n \\ m \text{ is even}}} \mu(m) 2^{\frac{n}{m}} \\
&= \sum_{\substack{m|n \\ m \text{ is odd}}} \mu(m) 2^{\frac{n}{m}} \\
&= -\sum_{\substack{2m|2n \\ m \text{ is odd}}} \mu(2m) 2^{\frac{2n}{2m}} \quad \text{since when } m \text{ is odd, } \mu(m) = -\mu(2m) \\
&= -\sum_{\substack{k|2n \\ k \text{ is even}}} \mu(k) 2^{\frac{2n}{k}} \quad \text{since when } 4|k,\ \mu(k) = 0 \\
&= \varepsilon_{2n}.
\end{aligned}$$

Second, assume $(\varepsilon'_n)_{n\geq 1}$ is a sequence satisfying recursion (1). Let $(u_n)_{n\geq 1}$ be the sequence defined by $u_n := \varepsilon'_n - \varepsilon_n$. Then, for $n \geq 1$, we have that

$$u_{2n-1} = \varepsilon'_{2n-1} - \varepsilon_{2n-1} = 0 \quad \text{and} \quad u_{2n} = \delta_n + \varepsilon'_n - \delta_n - \varepsilon_n = u_n.$$

It follows by induction that the sequence $(u_n)_{n\geq 1}$ identically vanishes, so that the sequence $(\varepsilon'_n)_{n\geq 1}$ is equal to the sequence $(\varepsilon_n)_{n\geq 1}$. □

It shall be convenient to consider three related sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ defined by

$$\alpha_n := \frac{\delta_n}{n} = \frac{1}{n}\sum_{m|n} \mu(m) 2^{\frac{n}{m}}, \quad \beta_n := \frac{\varepsilon_n}{n} = -\frac{1}{n} \sum_{\substack{m|n \\ m \text{ is even}}} \mu(m) 2^{\frac{n}{m}}$$

and

$$\gamma_n := \frac{\delta_n + \epsilon_n}{2n} = \frac{1}{2n} \sum_{\substack{m|n \\ m \text{ is odd}}} \mu(m) 2^{\frac{n}{m}}.$$

Lemma 2.4. *The sequences $(\beta_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ are characterized by the recursion*

$$\forall n \geq 1, \quad \beta_{2n-1} = 0 \quad \textit{and} \quad \beta_{2n} = \gamma_n = \frac{\alpha_n + \beta_n}{2}.$$

Proof. This is an immediate consequence of Lemma 2.1. □

3. Counting periodic sequences

3.1. **Symbolic dynamics.** Let us consider the set Σ of sequences of 0's and 1's:

$$\Sigma := \{0,1\}^{\mathbb{N}\smallsetminus\{0\}}.$$

A sequence in Σ shall be denoted by $\mathbf{s} = (s_n)_{n\geq 1}$. Consider the shift $\sigma : \Sigma \to \Sigma$ defined by

$$\sigma(s_1, s_2, s_3, \ldots) := (s_2, s_3, s_4, \ldots).$$

Let $\Theta \subset \Sigma$ be the subset of periodic sequences. Given $n \geq 1$, set

$$\widehat{\Theta}_n := \{\mathbf{s} \in \Sigma\ ;\ \sigma^{\circ n}(\mathbf{s}) = \mathbf{s}\}.$$

Note that $\widehat{\Theta}_n \subset \Theta$ is the set of sequences which are periodic under iteration of σ with period dividing n. Denote by $\Theta_n \subseteq \widehat{\Theta}_n$ the subset of sequences which have exact period n, so that

$$\Theta = \bigsqcup_{n\geq 1} \Theta_n.$$

Lemma 3.1. *For $n \geq 1$,* $\mathrm{card}(\Theta_n) = \delta_n$.

Proof. A sequence $\mathbf{s} \in \widehat{\Theta}_n$ is uniquely characterized by $(s_1, s_2, \ldots, s_n)$ which may be any element of $\{0,1\}^n$. As a consequence $\mathrm{card}(\widehat{\Theta}_n) = 2^n$. In addition,

$$\widehat{\Theta}_n = \bigsqcup_{m|n} \Theta_m \quad \text{so that} \quad 2^n = \sum_{m|n} \mathrm{card}(\Theta_m).$$

It follows from the Möbius inversion formula that for $n \geq 1$,

$$\mathrm{card}(\Theta_n) = \sum_{m|n} \mu(m) 2^{\frac{n}{m}} = \delta_n. \qquad \square$$

Consider now the involution $\iota : \Sigma \to \Sigma$ defined by

$$\iota(s_1, s_2, s_3, \ldots) := (1 - s_1, 1 - s_2, 1 - s_3, \ldots).$$

Note that σ and ι commute. A sequence $\mathbf{s} \in \Sigma$ is *reflexive* if its orbit under iteration of σ contains $\iota(\mathbf{s})$. Let $\Xi \subset \Sigma$ be the subset of reflexive sequences. A reflexive sequence is necessarily periodic since

$$\sigma^{\circ m}(\mathbf{s}) = \iota(\mathbf{s}) \quad \Rightarrow \quad \sigma^{\circ(2m)}(\mathbf{s}) = \mathbf{s}.$$

For $n \geq 1$, let Ξ_n be the set of reflexive sequences of period n:

$$\Xi_n := \Xi \cap \Theta_n.$$

Lemma 3.2. *For $n \geq 1$,* $\mathrm{card}(\Xi_n) = \varepsilon_n$.

Proof. Assume $\mathbf{s} \in \Xi_n$ and $\sigma^{\circ k}(\mathbf{s}) = \iota(\mathbf{s})$. Let $0 < m < n$ be congruent to k modulo n, so that $\sigma^{\circ m}(\mathbf{s}) = \sigma^{\circ k}(\mathbf{s}) = \iota(\mathbf{s})$. Note that $\sigma^{\circ(2m)}(\mathbf{s}) = \mathbf{s}$ so that n divides $2m$. Since $0 < 2m < 2n$, we have $n = 2m$. As a consequence, for $n \geq 1$,

$$\mathrm{card}(\Xi_{2n-1}) = 0.$$

For $n \geq 1$, set

$$\widehat{\Xi}_n := \{\mathbf{s} \in \Xi \; ; \; \sigma^{\circ n}(\mathbf{s}) = \mathbf{s}\} \quad \text{and} \quad \widehat{\Xi}'_n := \{\mathbf{s} \in \Xi \; ; \; \sigma^{\circ n}(\mathbf{s}) = \iota(\mathbf{s})\}.$$

Since $\iota : \Sigma \to \Sigma$ does not have any fixed point, $\widehat{\Xi}_n \cap \widehat{\Xi}'_n = \emptyset$. Assume $\mathbf{s} \in \widehat{\Xi}_{2n}$, i.e., $\mathbf{s}$ is a reflexive sequence of period dividing $2n$. Let $m \geq 1$ be the smallest integer such that $\sigma^{\circ m}(\mathbf{s}) = \iota(\mathbf{s})$. Then $\mathbf{s}$ has period $2m$ which divides $2n$, so that m divides n. If m divides n and n/m is odd, then $\mathbf{s} \in \widehat{\Xi}'_n$; and if n/m is even, then $\mathbf{s} \in \widehat{\Xi}_n$. Thus, for $n \geq 1$,

$$\widehat{\Xi}_{2n} = \widehat{\Xi}'_n \sqcup \widehat{\Xi}_n \quad \text{so that} \quad \mathrm{card}(\widehat{\Xi}_{2n}) = \mathrm{card}(\widehat{\Xi}'_n) + \mathrm{card}(\widehat{\Xi}_n). \tag{2}$$

A sequence $\mathbf{s} \in \widehat{\Xi}'_n$ is uniquely characterized by $(s_1, s_2, \ldots, s_n)$ which may be any element of $\{0,1\}^n$. Thus

$$\mathrm{card}(\widehat{\Xi}'_n) = 2^n = \sum_{m|n} \delta_m. \tag{3}$$

As in the previous proof,

$$\widehat{\Xi}_n = \bigsqcup_{m|n} \Xi_m \quad \text{so that} \quad \mathrm{card}(\widehat{\Xi}_n) = \sum_{m|n} \mathrm{card}(\Xi_m). \tag{4}$$

In addition, since $\mathrm{card}(\widehat{\Xi}_k) = 0$ if k is odd, we have that

$$\mathrm{card}(\widehat{\Xi}_{2n}) = \sum_{2m|2n} \mathrm{card}(\Xi_{2m}) = \sum_{m|n} \mathrm{card}(\Xi_{2m}). \tag{5}$$

Thus, for $n \geq 1$,

$$\begin{aligned} \sum_{m|n} \mathrm{card}(\Xi_{2m}) &= \mathrm{card}(\widehat{\Xi}_{2n}) && \text{from 5} \\ &= \mathrm{card}(\widehat{\Xi}'_n) + \mathrm{card}(\widehat{\Xi}_n) && \text{from 2} \\ &= \sum_{m|n} \delta_m + \sum_{m|n} \mathrm{card}(\Xi_m) && \text{from 3 and 4} \end{aligned}$$

so that for $n \geq 1$,

$$\mathrm{card}(\Xi_{2n}) = \delta_n + \mathrm{card}(\Xi_n).$$

The sequence $\big(\mathrm{card}(\Xi_n)\big)_{n\geq 1}$ satisfies recursion (1), thus is equal to $(\varepsilon_n)_{n\geq 1}$. □

3.2. **Multiplication by 2 in $\mathbb{R}/\mathbb{Z}$.** Consider the map $\tau : \Sigma \to \mathbb{R}/\mathbb{Z}$ defined by

$$\tau(\mathbf{s}) := \sum_{j\geq 1} \frac{s_j}{2^j} \pmod 1.$$

Note that $\tau : \Sigma \to \mathbb{R}/\mathbb{Z}$ is surjective (every angle in $\mathbb{R}/\mathbb{Z}$ has a binary expansion and $\mathbf{s}$ is the corresponding sequence of digits) but not injective. For example, $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ have the same image by τ. However, if two distinct sequences are identified, one is eventually constant equal to 0 and the other is eventually constant equal to 1. It follows that the only periodic sequences which are identified are $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$.

The map $\tau : \Sigma \to \mathbb{R}/\mathbb{Z}$ semi-conjugates the shift $\sigma : \Sigma \to \Sigma$ to the doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$D(\theta) := 2\theta,$$

that is, $D \circ \tau = \tau \circ \sigma$. An angle $\theta \in \mathbb{R}/\mathbb{Z}$ is periodic under iteration of D with period dividing n if and only if θ can be written as $m/(2^n - 1)$ with $m \in \mathbb{Z}/(2^n - 1)\mathbb{Z}$. In addition, if $\mathbf{s} \in \widehat{\Theta}_n$, i.e., if $\mathbf{s}$ is periodic with period dividing n, then

$$\tau(\mathbf{s}) = \frac{m}{2^n - 1} \quad \text{with} \quad m := \sum_{j=1}^{n} s_j 2^{n-j} \in \mathbb{Z}/(2^n - 1)\mathbb{Z}.$$

The map $\tau : \Sigma \to \mathbb{R}/\mathbb{Z}$ semi-conjugates the involution $\iota : \Sigma \to \Sigma$ to the involution $I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by

$$I(\theta) := -\theta,$$

that is, $I \circ \tau = \tau \circ \iota$. For $n = 1$, the doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has only 1 fixed point, namely 0, and this point is fixed by the involution I. For $n \geq 2$, it follows from §3.1 that the doubling map has δ_n periodic points of exact period n, and that among those, ε_n have an orbit which is invariant by the involution I. So, for $n \geq 2$, there are α_n orbits of period n and β_n among them are invariant by the involution I.

4. Quadratic dynamics in $\overline{\mathbb{Q}}$

In this section, we are mainly concerned with the dynamics of the quadratic polynomials $f_c : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ defined by

$$f_c(z) := z^2 + c \quad \text{with} \quad c \in \overline{\mathbb{Q}}.$$

For $c \in \overline{\mathbb{Q}}$, the periodic points of f_c of period dividing n are the roots of the polynomial $f_c^{\circ n}(z) - z \in \mathbb{Q}[z]$ which has degree 2^n. It shall therefore be convenient to consider the sequence $\big(F_n(c,z) \in \mathbb{Q}[c,z]\big)_{n\geq 1}$ of polynomials defined by

$$F_n(c,z) := f_c^{\circ n}(z) - z.$$

Those polynomials satisfy the recursion

$$F_1(c,z) = z^2 - z + c \quad \text{and} \quad F_{n+1}(c,z) = F_n(c,z)^2 + c.$$

It follows that they have integer coefficients, degree 2^{n-1} with respect to c and degree 2^n with respect to z. The coefficient of $c^{2^{n-1}}$ is 1 and the coefficient of z^{2^n} is 1.

4.1. **The dynamics of $f_0 : z \mapsto z^2$.** The periodic points of f_0 of period dividing n are the roots of the polynomial $F_n(0,z) = z^{2^n} - z \in \mathbb{Q}[z]$, whose roots are simple. The fixed points are 0 and 1. The periodic points of period $n \geq 2$ are roots of unity.

Let $\mathbb{U} \subset \overline{\mathbb{Q}}$ be the multiplicative group of roots of unity. The transcendental map $\mathbb{Q}/\mathbb{Z} \ni \theta \mapsto \exp(2\pi i\theta) \in \mathbb{U}$ is a group isomorphism which conjugates the doubling map $D : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ to the restriction $f_0 : \mathbb{U} \to \mathbb{U}$. It conjugates the involution $I : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ to the involution $\iota : \mathbb{U} \to \mathbb{U}$ defined by $\iota(z) = 1/z$.

It follows from §3.2 that for $n \geq 2$, the squaring map $f_0 : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ has δ_n periodic points of exact period n, and that among those, ε_n have an orbit which is invariant by the involution ι.

4.2. **The dynamics of $f_{-2} : z \mapsto z^2 - 2$.** Consider the map $\psi : \overline{\mathbb{Q}} \setminus \{0\} \to \overline{\mathbb{Q}}$ defined by

$$\psi(z) := z + \frac{1}{z}.$$

The map ψ is a ramified covering of degree 2. Each point in $\overline{\mathbb{Q}}$ has two distinct preimages in $\overline{\mathbb{Q}} \setminus \{0\}$ except 2 which has a single preimage at $z = 1$ and -2 which has a single preimage at $z = -1$. In addition,

$$\psi \circ f_0(z) = z^2 + \frac{1}{z^2} = \left(z + \frac{1}{z}\right)^2 - 2 = f_{-2} \circ \psi(z).$$

So, ψ semi-conjugates $f_0 : \overline{\mathbb{Q}} \setminus \{0\} \to \overline{\mathbb{Q}} \setminus \{0\}$ to $f_{-2} : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$.

4.3. **Real dynamics.** A parameter $c \in \overline{\mathbb{Q}}$ is a *center* if 0 is periodic under iteration of f_c. It is a *center of period* n if 0 has period n for f_c. The centers of period n are precisely the roots of the Gleason polynomial $G_n \in \mathbb{Q}[c]$ defined by

$$G_n(c) := \Phi_n(c, 0).$$

Example 4.1. We have that

$$G_1(c) = c, \quad G_2(c) = 1 + c, \quad G_3(c) = 1 + c + 2c^2 + c^3.$$

We shall say that c is a *real center* if $c \in \mathbb{R}$. The *kneading sequence* of a real center c is

$$\boldsymbol{\kappa}(c) := \big(\kappa_n\big)_{n\geq 0} \in \{+,-,\star\}^{\mathbb{N}} \quad \text{with} \quad \kappa_n = \begin{cases} + & \text{if } f_c^{\circ n}(0) > 0 \\ \star & \text{if } f_c^{\circ n}(0) = 0 \\ - & \text{if } f_c^{\circ n}(0) < 0 \end{cases}.$$

The *kneading angle* of a real center c is the angle $\boldsymbol{\theta}(c) := \tau\big(\boldsymbol{t}(c)\big) \in \mathbb{R}/\mathbb{Z}$ where $\boldsymbol{t}(c) := (t_n)_{n\geq 1} \in \Sigma$ is defined by

$$\forall n \geq 0 \quad t_{n+1} = \begin{cases} 0 & \text{if } \kappa_n = \star \\ t_n & \text{if } \kappa_n = + \\ 1 - t_n & \text{if } \kappa_n = - \end{cases}.$$

Example 4.2. The polynomial G_3 has a unique real root c_3. We have that

$$\boldsymbol{\kappa}(c_3) = (\star,-,+,\star,-,+,\ldots), \quad \boldsymbol{t}(c_3) = (0,1,1,0,1,1,\ldots) \quad \text{and} \quad \boldsymbol{\theta}(c_3) = \frac{3}{7}.$$

Note that by definition, the first digit in the binary expansion of $\boldsymbol{\theta}(c)$ is a 0, so that this angle belongs to the arc $[0, 1/2) \subset \mathbb{R}/\mathbb{Z}$. The following result is due to Milnor and Thurston [MT].

Theorem 4.3. *If $c_1 < c_2$ are real centers, then $\boldsymbol{\theta}(c_1) > \boldsymbol{\theta}(c_2)$. If c is a center of period n, then $\boldsymbol{\theta}(c)$ is periodic of period n for the doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. In addition, a periodic angle $\theta \in [0, 1/2) \subset \mathbb{R}/\mathbb{Z}$ of period n is the kneading angle of some real center of period n if and only if $\theta \in \mathbb{R}/\mathbb{Z}$ is the closest angle to $1/2 \in \mathbb{R}/\mathbb{Z}$ within its orbit under iteration of D.*

This result enables us to count the number of real centers of period n as follows. The doubling map has a unique orbit of period 1. This orbit is reduced to the angle $0 \in \mathbb{R}/\mathbb{Z}$ which is the angle of the unique center of period 1: $c = 0$. So assume the period is $n \geq 2$. On the one hand, assume O is an orbit for the doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is invariant by the involution $I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Then O contains exactly two angles closest to $1/2 \in \mathbb{R}/\mathbb{Z}$, one in the arc $(0, 1/2) \in \mathbb{R}/\mathbb{Z}$, the other in the arc $(-1/2, 0) \in \mathbb{R}/\mathbb{Z}$ being its image by the involution I. According to Theorem 4.3, there is exactly one real center with kneading angle in O. According to §3.2, there are β_n such orbits corresponding to β_n real centers of period n. On the other hand, assume O and O' are two distinct orbits of period n which are exchanged by the involution $I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. The closest angle to $1/2 \in \mathbb{R}/\mathbb{Z}$ in one of the two orbits is contained in the arc $(0, 1/2) \in \mathbb{R}/\mathbb{Z}$ and the closest angle to $1/2 \in \mathbb{R}/\mathbb{Z}$ in the other orbit is contained in the arc $(-1/2, 0) \in \mathbb{R}/\mathbb{Z}$. According to Theorem 4.3, there is exactly one real center with kneading angle in $O \cup O'$. There are $(\alpha_n - \beta_n)/2$ such pairs of orbits corresponding to $(\alpha_n - \beta_n)/2$ real centers of period n. Thus, the total number of real centers of period n is

$$\beta_n + \frac{\alpha_n - \beta_n}{2} = \frac{\alpha_n + \beta_n}{2} = \gamma_n = \frac{1}{2n} \sum_{\substack{m|n \\ m \text{ is odd}}} \mu(m) 2^{\frac{n}{m}}.$$

4.4. **Complex dynamics.** We shall prove Theorem 1.5 in this section. We have the following description of the kneading angle. Consider the family of quadratic polynomials $(f_c : \mathbb{C} \to \mathbb{C})_{c\in\mathbb{C}}$ defined by

$$f_c(z) := z^2 + c.$$

The Mandelbrot set $\mathcal{M}$ is the set of parameters c such that the orbit $\bigl(f_c^{\circ n}(0)\bigr)_{n\geq 0}$ is bounded. Douady and Hubbard proved that the Mandelbrot set is connected. More precisely, let $\mathbb{D}\subset\mathbb{C}$ be the unit disk. There exists a conformal isomorphism $\phi_{\mathcal{M}}:\mathbb{C}\smallsetminus\mathcal{M}\to\mathbb{C}\smallsetminus\overline{\mathbb{D}}$ which satisfies $\phi_{\mathcal{M}}(c)=c+\mathcal{O}(1)$ as $c\to\infty$. For $\theta\in\mathbb{R}/\mathbb{Z}$, the curve

$$\mathcal{R}(\theta):=\bigl\{c\in\mathbb{C}\smallsetminus\mathcal{M}\ ;\ \text{argument}\bigl(\phi_{\mathcal{M}}(c)\bigr)=2\pi\theta\ (\text{mod }2\pi)\bigr\}$$

is called the external ray of $\mathcal{M}$ of angle θ. If $\theta\in\mathbb{R}/\mathbb{Z}$ is periodic for the doubling map $D:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$, then the ray $\mathcal{R}(\theta)$ *lands* at a parameter $c\in\mathcal{M}$, i.e., $\overline{\mathcal{R}}(\theta)\cap\mathcal{M}=\{c\}$. Figure 1 shows the Mandelbrot set together with the rays of angle 1/3, 2/3, 3/7 and 4/7.

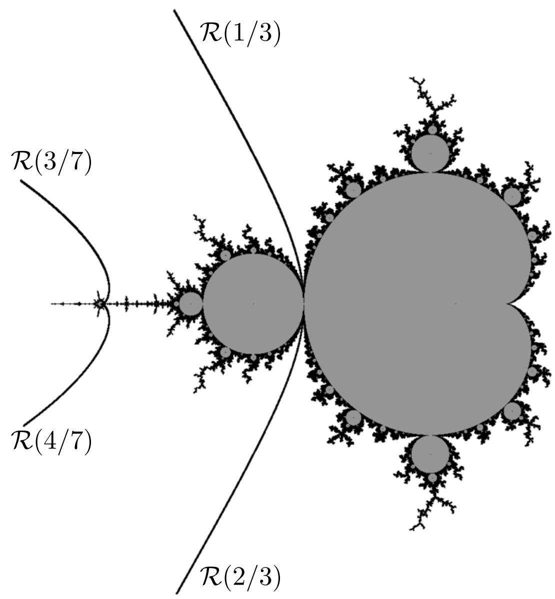

FIGURE 1. The Mandelbrot set. The external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ land at the root $c=-3/4$ of the satellite component $\mathcal{H}_{-1}$. The rays $\mathcal{R}(3/7)$ and $\mathcal{R}(4/7)$ land at the root $c=-7/4$ of the primitive component $\mathcal{H}_{c_3}$, where c_3 is the unique real center of period 3.

Assume now that c_0 is a center of period n. Then, c_0 is contained in the interior of $\mathcal{M}$. Let $\mathcal{H}_{c_0}$ be the connected component of the interior of the Mandelbrot set $\mathcal{M}$ containing c_0. Such a connected component $\mathcal{H}_{c_0}$ is called a *hyperbolic component of* $\mathcal{M}$. If $c\in\mathcal{H}_{c_0}$, the quadratic polynomial $f_c:z\mapsto z^2+c$ has an attracting cycle of period n. The product $\lambda(c)$ of the derivatives of f_c at the points of this cycle is called the *multiplier* of this cycle. To say that the cycle is attracting means that $\lambda(c)\in\mathbb{D}$. The map $\lambda:\mathcal{H}_{c_0}\to\mathbb{D}$ is a holomorphic isomorphism which extends as a homeomorphism $\lambda:\overline{\mathcal{H}}_{c_0}\to\overline{\mathbb{D}}$. The parameter $c_1:=\lambda^{-1}(1)$ is called the *root* of the hyperbolic component $\mathcal{H}_{c_0}$. The quadratic polynomial f_{c_1} has a *parabolic* cycle, i.e., a cycle whose multiplier is a root of unity. If this multiplier is 1, then $\mathcal{H}_{c_0}$ is called a *primitive* component of $\mathcal{M}$. Otherwise, $\mathcal{H}_{c_0}$ is called a *satellite* component of $\mathcal{M}$.

If $c_0 = 0$, which corresponds to the unique center of period 1, the root is $c_1 = 1/4$ and there is a single ray landing at c_1: the ray $\mathcal{R}(0)$. If c_0 is a center of period $n \geq 2$, there are two rays landing at c_1. When c_0 is real, then c_1 is also real and when the period is not 1, the two rays landing at c_1 are $\mathcal{R}\big(\boldsymbol{\theta}(c_0)\big)$ (which is contained in the upper half-plane) and $\mathcal{R}\big(-\boldsymbol{\theta}(c_0)\big)$ (which is contained in the lower half-plane). The angles $\boldsymbol{\theta}(c_0) \in \mathbb{R}/\mathbb{Z}$ and $-\boldsymbol{\theta}(c_0) \in \mathbb{R}/\mathbb{Z}$ belong to the same orbit under iteration of the doubling map $D : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ if and only if $\mathcal{H}_{c_0}$ is a satellite component of $\mathcal{M}$.

It follows from the count presented in §4.3 that among the γ_n real centers of period n, β_n are centers of satellite components of $\mathcal{M}$ and $\gamma_n - \beta_n$ are centers of primitive components of $\mathcal{M}$. Note that $\beta_n \neq 0$ only when n is even. In particular, if n is odd, the γ_n real centers of period n are centers of primitive components of $\mathcal{M}$. When n is even, $\beta_n = \gamma_{n/2}$. Thus, when n is even, among the γ_n centers of period n, there are $\gamma_{n/2}$ centers of satellite components and $\gamma_n - \gamma_{n/2}$ centers of primitive components. This completes the proof of Theorem 1.5.

4.5. **Lutzky's proof.** The original argument of Lutzky for counting the number of real centers may be illustrated by Figure 2.

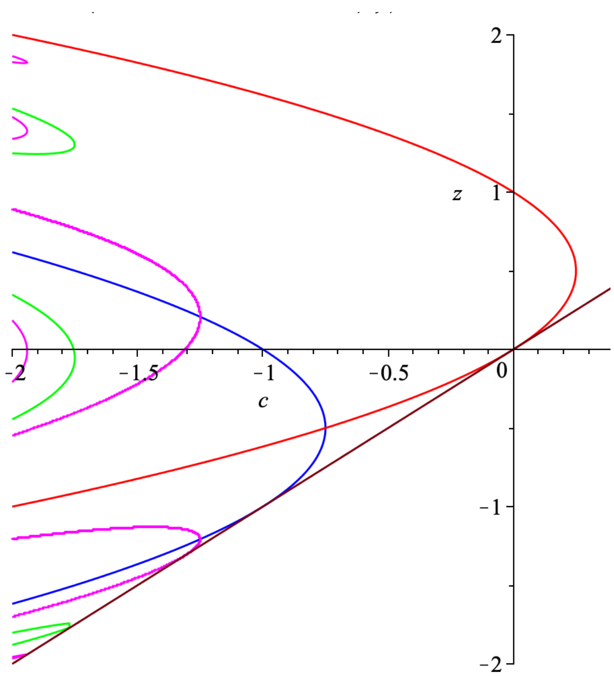

FIGURE 2. The curves of points $(c, z) \in [-2, 1/2] \times [-2, 2]$ such that z is periodic of period n for f_c. Red: $n = 1$; blue: $n = 2$, green: $n = 3$; pink: $n = 4$. The line of equation $z = c$ is tangent to those curves at points whose first coordinate is a real center.

For $c > 1/4$, the polynomial f_c has no real periodic point and for $c = -2$, the semi-conjugacy in §4.2 shows that the polynomial f_{-2} has α_n cycles of period n. As c increases from -2 to $1/4$, the α_n cycles must *bifurcate* in order to leave the real axis and become complex conjugate cycles. At a pitchfork bifurcation (which corresponds to roots of satellite components of period n), a single cycle bifurcates, contributing to one real center. At other bifurcations (which correspond to roots of primitive components), two cycles bifurcate, still contributing to only one real center. In addition, each pitchfork bifurcation comes from a bifurcation of period $n/2$. Thus, if γ'_n stands for the number of real centers of period n and β'_n stands for the number of pitchfork bifurcations of period n, then for $n \geq 1$,

$$\beta'_{2n-1} = 0, \quad \beta'_{2n} = \gamma'_n \quad \text{and} \quad \alpha_n = \beta'_n + 2(\gamma'_n - \beta'_n),$$

which may be re-written as

$$\beta'_{2n-1} = 0, \quad \beta'_{2n} = \gamma'_n = \frac{\alpha_n + \beta'_n}{2}.$$

According to Lemma 2.4, we have that $\beta'_n = \beta_n$ and $\gamma'_n = \gamma_n$ for $n \geq 1$ as required.

The justification that each bifurcation contributes to exactly one real center relies on the result of Milnor and Thurston stated previously.

5. Quadratic dynamics in $\overline{\mathbb{F}}_2$

In this section, we consider the case $\mathbf{k} = \mathbb{F}_2$. Theorem 1.6 will be established at the end of the section. Let us recall that for $n \geq 1$, the finite field $\mathbb{F}_{2^n}$ with 2^n elements is the splitting field of $z^{2^n} - z$ over $\mathbb{F}_2$. The Fröbenius endomorphism $f_0 : \overline{\mathbb{F}}_2 \to \overline{\mathbb{F}}_2$ is an automorphism of $\overline{\mathbb{F}}_2$ over $\mathbb{F}_2$: it fixes $\mathbb{F}_2$ pointwise and satisfies

$$f_0(z+w) = f_0(z) + f_0(w) \quad \text{and} \quad f_0(zw) = f_0(z)f_0(w).$$

More precisely, any point $z \in \overline{\mathbb{F}}_2$ is periodic for f_0. Such a point is periodic of period exactly n if and only if it is an element of $\mathbb{F}_{2^n}$ which is not contained in $\mathbb{F}_{2^m}$ for some proper positive divisor m of n. The conjugates of a point z of period n are the points of its orbit under iteration of f_0: z, $f_0(z)$, ..., $f_0^{\circ(n-1)}(z)$. This orbit is the *Galois orbit* of z. The minimal polynomial of such a point has degree n and vanishes precisely on its Galois orbit.

The periodic points of period n are the roots of the dynatomic polynomial

$$\phi_n(z) := \Phi_n(0, z) \in \mathbb{F}_2[z].$$

The irreducible monic polynomials of degree n in $\mathbb{F}_2[z]$ are the factors of ϕ_n. The polynomial ϕ_n has degree δ_n and simple roots (since it divides $z^{2^n} - z$ whose derivative is -1). So, there are precisely $\alpha_n = \delta_n/n$ monic irreducible polynomials of degree n in $\mathbb{F}_2[z]$. Equivalently, there are α_n Galois orbits of period n in $\overline{\mathbb{F}}_2$.

5.1. Critical orbit for f_c.

Lemma 5.1. *Assume $c \in \overline{\mathbb{F}}_2$. Then, for all $n \geq 1$,*

$$f_c^{\circ n}(0) = c_0 + c_1 + \ldots + c_{n-1} \quad \textit{with} \quad c_j := f_0^{\circ j}(c).$$

Proof. The proof goes by induction. For $n = 1$, we have that

$$f_c(0) = c = c_0.$$

And if

$$f_c^{\circ n}(0) = c_0 + c_1 + \ldots + c_{n-1},$$

then,

$$\begin{aligned} f_c^{\circ(n+1)}(0) &= c + f_0\big(f_c^{\circ n}(0)\big) \\ &= c + f_0(c_0) + f_0(c_1) + \cdots + f_0(c_{n-1}) = c_0 + c_1 + c_2 + \cdots + c_n. \end{aligned}$$ □

5.2. **Points in $\overline{\mathbb{F}}_2$ are centers.** We shall say that a Galois orbit is *centered* if the associated minimal polynomial is centered, and noncentered otherwise.

Let us recall that $c \in \overline{\mathbb{F}}_2$ is a *center* of period n if 0 is periodic of period n under iteration of f_c.

Lemma 5.2. *Any point $c \in \overline{\mathbb{F}}_2$ is a center. Let n be the period of c under iteration of f_0 and let m be the period of 0 under iteration of f_c. If the Galois orbit of c is centered, then $m = n$. If the Galois orbit of c is noncentered, then $m = 2n$.*

Proof. For $j \geq 0$, set

$$c_j := f_0^{\circ j}(c) \quad \text{and} \quad z_j := f_c^{\circ j}(0) = c_0 + c_1 + \ldots + c_{j-1}.$$

On the one hand, if c is periodic of period n for f_0, we have that $c_{n+j} = c_j$ for all $j \geq 0$, and so

$$z_{2n} = 2(c_0 + c_1 + \ldots + c_{n-1}) = 0.$$

Thus, 0 is periodic for f_c and the period m divides $2n$. On the other hand, if $z_m = 0$ for some $m \geq 1$, then

$$f_0^{\circ m}(c) = c_m = z_m + c_m = z_{m+1} = z_m^2 + c = c.$$

Thus, the period n of c for f_0 divides m. Since m divides $2n$ and n divides m, this forces either $m = n$ or $m = 2n$.

Let P be the minimal polynomial of c. Its roots are the points $c_0, c_1, \ldots, c_{n-1}$. As a consequence, 0 is periodic of period n for f_c if and only if $z_n = 0$, i.e., if and only if $c_0 + c_1 + \ldots c_{n-1} = 0$, i.e., if and only if P is centered. □

5.3. **From dynamical plane to parameter space.** Let $\iota : \overline{\mathbb{F}}_2 \smallsetminus \{0\} \to \overline{\mathbb{F}}_2 \smallsetminus \{0\}$ be the involution defined by

$$\iota(\vartheta) = \frac{1}{\vartheta}.$$

Assume $\vartheta \in \overline{\mathbb{F}}_2 \smallsetminus \{0\}$. Then,

$$\vartheta + \iota(\vartheta) = 0 \quad \Leftrightarrow \quad \vartheta^2 + 1 = 0 \quad \Leftrightarrow \quad \vartheta = 1.$$

We may therefore consider the map $\psi : \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2 \to \overline{\mathbb{F}}_2 \smallsetminus \{0\}$ defined by

$$\psi(\vartheta) := \frac{1}{\vartheta + \iota(\vartheta)} = \frac{\vartheta}{\vartheta^2 + 1}.$$

The involution ι and the map ψ commute with f_0. So, they send Galois orbits to Galois orbits.

Lemma 5.3. *The map $\psi : \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2 \to \overline{\mathbb{F}}_2 \smallsetminus \{0\}$ is surjective and each fiber contains two distinct points; those are exchanged by the involution ι.*

Proof. Assume $c \in \overline{\mathbb{F}}_2 \smallsetminus \{0\}$. Then, $\psi(\vartheta) = c$ if and only if $\vartheta^2 - \vartheta/c + 1 = 0$. The discriminant of this quadratic polynomial is $1/c^2 \neq 0$. So, there are two distinct roots. The product of the roots is 1. So, they are exchanged by ι. □

Lemma 5.4. *Assume $\vartheta \in \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2$ and $c = \psi(\vartheta)$. Let n be the period of ϑ for f_0 and let m be the period of c for f_0. If ϑ is conjugate to $1/\vartheta$, then $n = 2m$ and the minimal polynomial of c is noncentered. Otherwise $n = m$ and the minimal polynomial of c is centered.*

Proof. By assumption, the Galois orbit of ϑ contains n points and the Galois orbit of c contains m points. Since ψ commutes with f_0, it sends the Galois orbit of ϑ to the Galois orbit of c. According to Lemma 5.3, the fibers of $\psi : \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2 \to \overline{\mathbb{F}}_2 \smallsetminus \{0\}$ contain exactly two points which are exchanged by the involution ι. So, if the Galois orbit of ϑ is preserved by the involution ι, then its image by ψ contains $m = n/2$ points. Otherwise it contains $m = n$ points. □

Lemma 5.5. *Assume $\vartheta \in \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2$ and $c = \psi(\vartheta)$. Then, for $n \geq 1$,*

$$f_c^{\circ n}(0) = \frac{\vartheta + \vartheta^2 + \vartheta^3 + \cdots + \vartheta^{2^n-1}}{\vartheta^{2^n}+1}.$$

Proof. For $n \geq 0$, set

$$c_n := f_0^{\circ n}(c) = \frac{\vartheta^{2^n}}{\vartheta^{2^{n+1}}+1} = \frac{\vartheta^{2^n}}{(\vartheta^{2^n}+1)^2}.$$

According to Lemma 5.1, we have that for $n \geq 0$,

$$f_c^{\circ n}(0) = c_0 + c_1 + \cdots + c_{n-1}.$$

Now, the proof goes by induction. For $n = 1$, we have that

$$f_c(0) = c = \frac{1}{\vartheta + 1/\vartheta} = \frac{\vartheta}{\vartheta^2+1}.$$

And if

$$f_c^{\circ n}(0) = \frac{\vartheta + \vartheta^2 + \vartheta^3 + \cdots + \vartheta^{2^n-1}}{\vartheta^{2^n}+1},$$

then

$$\begin{aligned} f_c^{\circ(n+1)}(0) &= f_c^{\circ n}(0) + c_n \\ &= \frac{\vartheta + \vartheta^2 + \cdots + \vartheta^{2^n-1}}{\vartheta^{2^n}+1} + \frac{\vartheta^{2^n}}{(1+\vartheta^{2^n})(\vartheta^{2^n}+1)} \\ &= \frac{(\vartheta + \vartheta^2 + \cdots + \vartheta^{2^n-1})(1+\vartheta^{2^n}) + \vartheta^{2^n}}{(1+\vartheta^{2^n})(\vartheta^{2^n}+1)} \\ &= \frac{(\vartheta + \cdots + \vartheta^{2^n-1}) + \vartheta^{2^n} + (\vartheta^{2^n+1} + \cdots + \vartheta^{2^{n+1}-1})}{\vartheta^{2^{n+1}}+1}. \quad \square \end{aligned}$$

Lemma 5.6. *Assume $\vartheta \in \overline{\mathbb{F}}_2 \smallsetminus \mathbb{F}_2$ is periodic of period $n \geq 2$ for f_0 and $c = \psi(\vartheta)$. Then, 0 is periodic of period n for f_c.*

Proof. According to Lemma 5.5, for $j \geq 1$,

$$f_c^{\circ j}(0) = \frac{\vartheta + \vartheta^2 + \vartheta^3 + \cdots + \vartheta^{2^j-1}}{\vartheta^{2^j}+1} = \frac{\vartheta^{2^j} - \vartheta}{(\vartheta-1)(\vartheta^{2^j}+1)} = \frac{f_0^{\circ j}(\vartheta) - \vartheta}{(\vartheta-1)^{2^j+1}}.$$

Thus, $f_c^{\circ j}(0) = 0$ if and only if j is a multiple of n. □

5.4. **Counting orbits.** For $n \geq 1$, let α'_n be the number of Galois orbits in $\overline{\mathbb{F}}_2 \smallsetminus \{0\}$ which have period n, and let β'_n be the number of those orbits which are invariant by the involution ι. Then, $\alpha'_1 = \beta'_1 = 1$ since the only fixed point of f_0 in $\overline{\mathbb{F}}_2 \smallsetminus \{0\}$ is 1. And $\alpha'_n = \alpha_n$ for $n \geq 2$.

Lemma 5.7. *We have that*

$$\beta'_1 = 1 \quad \textit{and} \quad \forall n \geq 1 \quad \begin{cases} \beta'_{2n} = \dfrac{\alpha'_n + \beta'_n}{2} \\ \beta'_{2n+1} = 0. \end{cases}$$

Proof. The only fixed point of ι is 1, which is a fixed point of f_0. So, $\beta'_1 = 1$ and if a Galois orbit is preserved by ι, then its cardinality must be even. It follows that $\beta'_{2n+1} = 0$. Next, a Galois orbit of period n for f_0 is the image by ψ of

- either a Galois orbit of period $2n$ which is invariant by ι,
- or two distinct Galois orbits of period n which are exchanged by ι.

It follows that

$$\alpha'_n = \beta'_{2n} + \frac{\alpha'_n - \beta'_n}{2} \quad \text{so that} \quad \beta'_{2n} = \frac{\alpha'_n + \beta'_n}{2}. \qquad \square$$

Lemma 5.8. *We have that $\beta'_n = \beta_n$ for $n \geq 2$.*

Proof. Consider the sequence $(\beta''_n)_{n\geq 1}$ defined by

$$\beta''_1 := 0 \quad \text{and} \quad \forall n \geq 2 \quad \beta''_n := \beta'_n.$$

Note that $\alpha'_1 + \beta'_1 = 2 = \alpha_1 + \beta''_1$, so that for $n \geq 1$,

$$\alpha'_n + \beta'_n = \alpha_n + \beta''_n.$$

Thus, according to Lemma 5.7,

$$\forall n \geq 1, \quad \beta''_{2n-1} = 0 \quad \text{and} \quad \beta''_{2n} = \beta'_{2n} = \frac{\alpha'_n + \beta'_n}{2} = \frac{\alpha_n + \beta''_n}{2}.$$

According to Lemma 2.4, we have that $\beta''_n = \beta_n$ for $n \geq 1$. $\square$

Lemma 5.9. *For $n \geq 1$, the n-th Gleason polynomial has γ_n monic irreducible factors in $\mathbb{F}_2[c]$.*

Proof. For $n \geq 1$, let γ'_n be the number of Galois orbits of centers of period n in $\overline{\mathbb{F}}_2$. For $n = 1$, we have $\gamma'_1 = 1 = \gamma_1$. For $n \geq 2$, according to Lemma 5.6, the Galois orbits of centers of period n are the images by ψ of the Galois orbits of period n for f_0. According to Lemma 5.4, the centered ones are the images of the Galois orbits which are not invariant by the involution ι. There are $(\alpha_n - \beta_n)/2$ such orbits. The noncentered ones are the images of the Galois orbits which are invariant by the involution ι. There are β_n such orbits. Therefore,

$$\gamma'_n = \frac{\alpha_n - \beta_n}{2} + \beta_n = \frac{\alpha_n + \beta_n}{2} = \gamma_n. \qquad \square$$

This completes the proof of Theorem 1.6

6. Dynatomic polynomials in $\mathbb{F}_2[c, z]$

We finally prove Theorem 1.2. The proof relies on the following observation. Recall that for $n \geq 1$,

$$F_n(c, z) := f_c^{\circ n}(z) - z.$$

Lemma 6.1. *For $n \geq 1$, we have the following equality in $\mathbb{F}_2[c, z]$:*

$$F_n(c, z) = H_n(z^2 + c - z) \quad \textit{with} \quad H_n(c) := F_n(c, 0).$$

Proof. Observe that for $n \geq 1$,

$$H_{n+1}(c) = f_c^{\circ(n+1)}(0) = \big(f_c^{\circ n}(0)\big)^2 + c = H_n^2(c) + c.$$

We shall prove the result by induction on $n \geq 1$. For $n = 1$, we have that

$$F_1(c, z) = f_c(z) - z = z^2 + c - z.$$

So, the result holds.

Let us now assume that for some $n \geq 1$,

$$F_n(c, z) = H_n(z^2 + c - z).$$

Then,

$$\begin{aligned} F_{n+1}(c, z) &= \big(F_n(c, z) + z\big)^2 + c - z \\ &= \big(H_n(z^2 + c - z) + z\big)^2 + c - z \\ &= H_n^2(z^2 + c - z) + z^2 + c - z = H_{n+1}(z^2 + c - z). \end{aligned}$$

This completes the proof by induction. □

For $n \geq 1$, we now have

$$F_n(c, z) = \prod_{m|n} \Phi_m(c, z) \quad \text{and} \quad H_n(c) = \prod_{m|n} G_m(c).$$

As a consequence, for $n \geq 1$,

$$\Phi_n(c, z) = G_n(z^2 + c - z).$$

On the one hand, it follows that if $P(c)$ divides $G_n(c)$, then $P(z^2 + c - z)$ divides $\Phi_n(c, z)$. Thus, Φ_n has at least γ_n irreducible factors which are monic with respect to c. On the other hand, if $Q(c, z)$ is a factor of $\Phi_n(c, z)$ which is monic with respect to c, then $Q(c, 0)$ is a monic factor of $G_n(c)$. This shows that $\Phi_n(c, z)$ has at most γ_n factors which are monic with respect to c. Thus, $\Phi_n(c, z)$ has exactly γ_n factors which are monic with respect to c. Theorem 1.2 now follows easily from Theorem 1.6.

Appendix A. Itineraries of roots of low-degree Gleason polynomials

In this appendix, we present for each period $n \in [1, 8]$ two tables. The first table corresponds to $\mathbf{k} = \mathbb{Q}$. It contains:

- the (approximate) value of the real center of period n,
- the initial segment of its kneading sequence (to be repeated periodically with period n),
- the kneading angle $\boldsymbol{\theta}(c)$ with its binary expansion and
- the cycles in $\mathbb{Z}/(2^n - 1)\mathbb{Z}$ of $(2^n - 1)\boldsymbol{\theta}(c)$ and $-(2^n - 1)\boldsymbol{\theta}(c)$.

The second table corresponds to $\mathbf{k} = \mathbb{F}_2$. It contains:

- the minimal polynomials $P \in \mathbb{F}_2[c]$ of the centers of period n,
- the coefficients of c^k of $P(c)$ and
- the minimal polynomials of the numbers $\vartheta \in \overline{\mathbb{F}}_2$ such that $P \circ \psi(\vartheta) = 0$.

For periods 9 and 10, we only present the first table.

A.1. **Period 1.**

$$G_1(c) = c \quad \text{and} \quad \gamma_1 = 1.$$

0	$(\star)$	$0/1 = .\overline{0}$	$\{0\}$

c	$(0,1)$	ϑ

A.2. **Period 2.**

$$G_2(c) = 1 + c \quad \text{and} \quad \gamma_2 = 1.$$

-1	$(\star, -)$	$1/3 = .\overline{01}$	$\{1, 2\}$

$1 + c$	$(1,1)$	$1 + \vartheta + \vartheta^2$

A.3. **Period 3.**

$$G_3(c) = 1 + c + 2c^2 + c^3 \quad \text{and} \quad \gamma_3 = 1.$$

-1.754878	$(\star, -, +)$	$3/7 = .\overline{011}$	$\{1, 2, 4\} \quad \{6, 5, 3\}$

$1 + c + c^3$	$(1,1,0,1)$	$(1 + \vartheta + \vartheta^3)(\vartheta^3 + \vartheta^2 + 1)$

A.4. **Period 4.**

$$G_4(c) = 1 + 2c^2 + 3c^3 + 3c^4 + 3c^5 + c^6 \quad \text{and} \quad \gamma_4 = 2.$$

-1.940800	$(\star, -, +, +)$	$7/15 = .\overline{0111}$	$\{1, 2, 4, 8\} \quad \{14, 13, 11, 7\}$
-1.310703	$(\star, -, +, -)$	$6/15 = 2/5 = .\overline{0110}$	$\{3, 6, 12, 9\}$

$1 + c + c^4$	$(1,1,0,0,1)$	$(1 + \vartheta + \vartheta^4)(\vartheta^4 + \vartheta^3 + 1)$
$1 + c + c^2$	$(1,1,1)$	$1 + \vartheta + \vartheta^2 + \vartheta^3 + \vartheta^4$

A.5. **Period 5.**

$$\begin{aligned} G_5(c) \quad = \quad & 1 + c + 2c^2 + 5c^3 + 14c^4 + 26c^5 + 44c^6 + 69c^7 + 94c^8 \\ & +114c^9 + 116c^{10} + 94c^{11} + 60c^{12} + 28c^{13} + 8c^{14} + c^{15} \end{aligned}$$

and

$$\gamma_5 = 3.$$

-1.985424	$(\star, -, +, +, +)$	$15/31 = .\overline{01111}$	$\{1, 2, 4, 8, 16\} \quad \{30, 29, 27, 23, 15\}$
-1.860783	$(\star, -, +, +, -)$	$14/31 = .\overline{01110}$	$\{3, 6, 12, 24, 17\} \quad \{28, 25, 19, 7, 14\}$
-1.625414	$(\star, -, +, -, -)$	$13/31 = .\overline{01101}$	$\{5, 10, 20, 9, 18\} \quad \{26, 21, 11, 22, 13\}$

$1 + c^2 + c^5$	$(1,0,1,0,0,1)$	$(1 + \vartheta + \vartheta^2 + \vartheta^4 + \vartheta^5)(\vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta + 1)$
$1 + c^3 + c^5$	$(1,0,0,1,0,1)$	$(1 + \vartheta^3 + \vartheta^5)(\vartheta^5 + \vartheta^2 + 1)$
$1 + c + c^2 + c^3 + c^5$	$(1,1,1,1,0,1)$	$(1 + \vartheta + \vartheta^2 + \vartheta^3 + \vartheta^5)(\vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta^2 + 1)$

A.6. **Period 6.**

$$\begin{aligned} G_6(c) &= 1 - c + c^2 + 3c^3 + 7c^4 + 17c^5 + 35c^6 + 76c^7 + 155c^8 + 298c^9 \\ &\quad +536c^{10} + 927c^{11} + 1525c^{12} + 2331c^{13} + 3310c^{14} + 4346c^{15} \\ &\quad +5258c^{16} + 5843c^{17} + 5892c^{18} + 5313c^{19} + 4219c^{20} + 2892c^{21} \\ &\quad +1672c^{22} + 792c^{23} + 293c^{24} + 78c^{25} + 13c^{26} + c^{27} \end{aligned}$$

and

$$\gamma_6 = 5.$$

-1.996376	$(\star,-,+,+,+,+)$	$31/63 = .\overline{011111}$	$\{1,2,4,8,16,32\}$ $\{62,61,59,55,47,31\}$
-1.966773	$(\star,-,+,+,+,-)$	$30/63 = 10/21 = .\overline{011110}$	$\{3,6,12,24,48,33\}$ $\{60,57,51,39,15,30\}$
-1.907280	$(\star,-,+,+,-,-)$	$29/63 = .\overline{011101}$	$\{5,10,20,40,17,34\}$ $\{58,53,43,23,46,29\}$
-1.772893	$(\star,-,+,+,-,+)$	$28/63 = 4/9 = .\overline{011100}$	$\{7,14,28,56,49,35\}$
-1.476015	$(\star,-,+,-,-,-)$	$26/63 = .\overline{011010}$	$\{11,22,44,25,50,37\}$ $\{52,41,19,38,13,26\}$

$1+c^2+c^3$	$(1,0,1,1)$	$1+\vartheta^3+\vartheta^6$
$1+c+c^6$	$(1,1,0,0,0,0,1)$	$(1+\vartheta+\vartheta^2+\vartheta^4+\vartheta^6)(\vartheta^6+\vartheta^5+\vartheta^4+\vartheta^2+1)$
$1+c^3+c^6$	$(1,0,0,1,0,0,1)$	$(1+\vartheta+\vartheta^2+\vartheta^5+\vartheta^6)(\vartheta^6+\vartheta^5+\vartheta^4+\vartheta+1)$
$1+c+c^2+c^4+c^6$	$(1,1,1,0,1,0,1)$	$(1+\vartheta+\vartheta^3+\vartheta^4+\vartheta^6)(\vartheta^6+\vartheta^5+\vartheta^3+\vartheta^2+1)$
$1+c+c^3+c^4+c^6$	$(1,1,0,1,1,0,1)$	$(1+\vartheta^5+\vartheta^6)(\vartheta^6+\vartheta+1)$

A.7. **Period 7.**

$$\deg(G_7) = 63 \quad \text{and} \quad \gamma_7 = 9.$$

-1.999096	$(\star,-,+,+,+,+,+)$	$\frac{63}{127} = .\overline{0111111}$	$\{1,2,4,8,16,32,64\}$ $\{126,125,123,119,111,95,63\}$
-1.991814	$(\star,-,+,+,+,+,-)$	$\frac{62}{127} = .\overline{0111110}$	$\{3,6,12,24,48,96,65\}$ $\{124,121,115,103,79,31,62\}$
-1.977180	$(\star,-,+,+,+,-,-)$	$\frac{61}{127} = .\overline{0111101}$	$\{5,10,20,40,80,33,66\}$ $\{122,117,107,87,47,94,61\}$
-1.953706	$(\star,-,+,+,+,-,+)$	$\frac{60}{127} = .\overline{0111100}$	$\{7,14,28,56,112,97,67\}$ $\{120,113,99,71,15,30,60\}$
-1.927148	$(\star,-,+,+,-,-,+)$	$\frac{59}{127} = .\overline{0111011}$	$\{9,18,36,72,17,34,68\}$ $\{118,109,91,55,110,93,59\}$
-1.884804	$(\star,-,+,+,-,-,-)$	$\frac{58}{127} = .\overline{0111010}$	$\{11,22,44,88,49,98,69\}$ $\{116,105,83,39,78,29,58\}$
-1.832315	$(\star,-,+,+,-,+,-)$	$\frac{57}{127} = .\overline{0111001}$	$\{13,26,52,104,81,35,70\}$ $\{114,101,75,23,46,92,57\}$
-1.674066	$(\star,-,+,-,-,+,-)$	$\frac{54}{127} = .\overline{0110110}$	$\{19,38,76,25,50,100,73\}$ $\{108,89,51,102,77,27,54\}$
-1.574889	$(\star,-,+,-,-,-,-)$	$\frac{53}{127} = .\overline{0110101}$	$\{21,42,84,41,82,37,74\}$ $\{106,85,43,86,45,90,53\}$

$1+c+c^7$	$(1,1,0,0,0,0,0,1)$	$(\vartheta^7+\vartheta^5+\vartheta^3+\vartheta+1)(\vartheta^7+\vartheta^6+\vartheta^4+\vartheta^2+1)$
$1+c^3+c^7$	$(1,0,0,1,0,0,0,1)$	$(\vartheta^7+\vartheta^6+\vartheta^5+\vartheta^3+\vartheta^2+\vartheta+1)(\vartheta^7+\vartheta^6+\vartheta^5+\vartheta^4+\vartheta^2+\vartheta+1)$
$1+c+c^2+c^3+c^7$	$(1,1,1,1,0,0,0,1)$	$(\vartheta^7+\vartheta^5+\vartheta^4+\vartheta^3+\vartheta^2+\vartheta+1)(\vartheta^7+\vartheta^6+\vartheta^5+\vartheta^4+\vartheta^3+\vartheta^2+1)$
$1+c^4+c^7$	$(1,0,0,0,1,0,0,1)$	$(\vartheta^7+\vartheta^5+\vartheta^4+\vartheta^3+1)(\vartheta^7+\vartheta^4+\vartheta^3+\vartheta^2+1)$
$1+c^2+c^3+c^4+c^7$	$(1,0,1,1,1,0,0,1)$	$(\vartheta^7+\vartheta^4+1)(\vartheta^7+\vartheta^3+1)$
$1+c+c^2+c^5+c^7$	$(1,1,1,0,0,1,0,1)$	$(\vartheta^7+\vartheta^6+1)(\vartheta^7+\vartheta+1)$
$1+c+c^3+c^5+c^7$	$(1,1,0,1,0,1,0,1)$	$(\vartheta^7+\vartheta^3+\vartheta^2+\vartheta+1)(\vartheta^7+\vartheta^6+\vartheta^5+\vartheta^4+1)$
$1+c^3+c^4+c^5+c^7$	$(1,0,0,1,1,1,0,1)$	$(\vartheta^7+\vartheta^6+\vartheta^3+\vartheta+1)(\vartheta^7+\vartheta^6+\vartheta^4+\vartheta+1)$
$1+c+c^2+c^3+c^4+c^5+c^7$	$(1,1,1,1,1,1,0,1)$	$(\vartheta^7+\vartheta^6+\vartheta^5+\vartheta^2+1)(\vartheta^7+\vartheta^5+\vartheta^2+\vartheta+1)$

A.8. **Period 8.**

$$\deg(G_8) = 120 \quad \text{and} \quad \gamma_8 = 16.$$

-1.999774	$(\star, -, +, +, +, +, +, +)$	$\frac{127}{255} = .\overline{01111111}$	$\{1, 2, 4, 8, 16, 32, 64, 128\}$ $\{254, 253, 251, 247, 239, 223, 191, 127\}$
-1.997963	$(\star, -, +, +, +, +, +, -)$	$\frac{126}{255} = .\overline{01111110}$	$\{3, 6, 12, 24, 48, 96, 192, 129\}$ $\{252, 249, 243, 231, 207, 159, 63, 126\}$
-1.994333	$(\star, -, +, +, +, +, -, -)$	$\frac{125}{255} = .\overline{01111101}$	$\{5, 10, 20, 40, 80, 160, 65, 130\}$ $\{250, 245, 235, 215, 175, 95, 190, 125\}$
-1.988793	$(\star, -, +, +, +, +, -, +)$	$\frac{124}{255} = .\overline{01111100}$	$\{7, 14, 28, 56, 112, 224, 193, 131\}$ $\{248, 241, 227, 199, 143, 31, 62, 124\}$
-1.981656	$(\star, -, +, +, +, -, -, +)$	$\frac{123}{255} = .\overline{01111011}$	$\{9, 18, 36, 72, 144, 33, 66, 132\}$ $\{246, 237, 219, 183, 111, 222, 189, 123\}$
-1.972200	$(\star, -, +, +, +, -, -, -)$	$\frac{122}{255} = .\overline{01111010}$	$\{11, 22, 44, 88, 176, 97, 194, 133\}$ $\{244, 233, 211, 167, 79, 158, 61, 122\}$
-1.960759	$(\star, -, +, +, +, -, +, -)$	$\frac{121}{255} = .\overline{01111001}$	$\{13, 26, 52, 104, 208, 161, 67, 134\}$ $\{242, 229, 203, 151, 47, 94, 188, 121\}$
-1.941782	$(\star, -, +, +, +, -, +, +)$	$\frac{120}{255} = \frac{8}{17} = .\overline{01111000}$	$\{15, 30, 60, 120, 240, 225, 195, 135\}$
-1.917098	$(\star, -, +, +, -, -, +, -)$	$\frac{118}{255} = .\overline{01110110}$	$\{19, 38, 76, 152, 49, 98, 196, 137\}$ $\{236, 217, 179, 103, 206, 157, 59, 118\}$
-1.896918	$(\star, -, +, +, -, -, -, -)$	$\frac{117}{255} = .\overline{01110101}$	$\{21, 42, 84, 168, 81, 162, 69, 138\}$ $\{234, 213, 171, 87, 174, 93, 186, 117\}$
-1.870004	$(\star, -, +, +, -, -, -, +)$	$\frac{116}{255} = .\overline{01110100}$	$\{23, 46, 92, 184, 113, 226, 197, 139\}$ $\{232, 209, 163, 71, 142, 29, 58, 116\}$
-1.851730	$(\star, -, +, +, -, +, -, +)$	$\frac{115}{255} = .\overline{01110011}$	$\{25, 50, 100, 200, 145, 35, 70, 140\}$ $\{230, 205, 155, 55, 110, 220, 185, 115\}$
-1.810001	$(\star, -, +, +, -, +, -, -)$	$\frac{114}{255} = .\overline{01110010}$	$\{27, 54, 108, 216, 177, 99, 198, 141\}$ $\{228, 201, 147, 39, 78, 156, 57, 114\}$
-1.711079	$(\star, -, +, -, -, +, -, -)$	$\frac{109}{255} = .\overline{01101101}$	$\{37, 74, 148, 41, 82, 164, 73, 146\}$ $\{218, 181, 107, 214, 173, 91, 182, 109\}$
-1.521817	$(\star, -, +, -, -, -, -, -)$	$\frac{106}{255} = .\overline{01101010}$	$\{43, 86, 172, 89, 178, 101, 202, 149\}$ $\{212, 169, 83, 166, 77, 154, 53, 106\}$
-1.381547	$(\star, -, +, -, -, -, +, -)$	$\frac{105}{255} = \frac{7}{17} = .\overline{01101001}$	$\{45, 90, 180, 105, 210, 165, 75, 150\}$

$1 + c^3 + c^4$	$(1, 0, 0, 1, 1)$	$\vartheta^8 + \vartheta^5 + \vartheta^4 + \vartheta^3 + 1$
$1 + c + c^2 + c^3 + c^4$	$(1, 1, 1, 1, 1)$	$\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^4 + \vartheta^2 + \vartheta + 1$
$1 + c + c^3 + c^4 + c^8$	$(1, 1, 0, 1, 1, 0, 0, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^5 + \vartheta^3 + 1)(\vartheta^8 + \vartheta^5 + \vartheta^3 + \vartheta + 1)$
$1 + c^2 + c^3 + c^4 + c^8$	$(1, 0, 1, 1, 1, 0, 0, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^5 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^3 + \vartheta + 1)$
$1 + c + c^3 + c^5 + c^8$	$(1, 1, 0, 1, 0, 1, 0, 0, 1)$	$(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta^2 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^4 + \vartheta^3 + \vartheta^2 + 1)$
$1 + c^2 + c^3 + c^5 + c^8$	$(1, 0, 1, 1, 0, 1, 0, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^5 + \vartheta^2 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^3 + \vartheta^2 + \vartheta + 1)$
$1 + c^3 + c^4 + c^5 + c^8$	$(1, 0, 0, 1, 1, 1, 0, 0, 1)$	$(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta^2 + 1)(\vartheta^8 + \vartheta^6 + \vartheta^3 + \vartheta^2 + 1)$
$1 + c + c^2 + c^3 + c^4 + c^5 + c^8$	$(1, 1, 1, 1, 1, 1, 0, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^3 + \vartheta^2 + 1)(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta + 1)$
$1 + c^2 + c^3 + c^6 + c^8$	$(1, 1, 1, 1, 0, 0, 1, 0, 1)$	$(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta^4 + 1)(\vartheta^8 + \vartheta^4 + \vartheta^3 + \vartheta^2 + 1)$
$1 + c + c^2 + c^3 + c^4 + c^6 + c^8$	$(1, 1, 1, 1, 1, 0, 1, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta^3 + 1)(\vartheta^8 + \vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta^2 + \vartheta + 1)$
$1 + c + c^5 + c^6 + c^8$	$(1, 1, 0, 0, 0, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^4 + \vartheta^3 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^5 + \vartheta^4 + 1)$
$1 + c^2 + c^5 + c^6 + c^8$	$(1, 0, 1, 0, 0, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^5 + \vartheta^3 + \vartheta^2 + 1)(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta^3 + 1)$
$1 + c^3 + c^5 + c^6 + c^8$	$(1, 0, 0, 1, 0, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^2 + \vartheta + 1)$
$1 + c^4 + c^5 + c^6 + c^8$	$(1, 0, 0, 0, 1, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^7 + \vartheta^4 + \vartheta^3 + \vartheta^2 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta + 1)$
$1 + c + c^2 + c^4 + c^5 + c^6 + c^8$	$(1, 1, 1, 0, 1, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^5 + \vartheta^4 + \vartheta^3 + \vartheta^2 + 1)$
$1 + c + c^3 + c^4 + c^5 + c^6 + c^8$	$(1, 1, 0, 1, 1, 1, 1, 0, 1)$	$(\vartheta^8 + \vartheta^6 + \vartheta^4 + \vartheta^3 + \vartheta^2 + \vartheta + 1)(\vartheta^8 + \vartheta^7 + \vartheta^6 + \vartheta^5 + \vartheta^4 + \vartheta^2 + 1)$

A.9. **Period 9.**

$$\deg(G_9) = 252 \quad \text{and} \quad \gamma_9 = 28.$$

-1.999944	$(\star,-,+,+,+,+,+,+,+)$	$\frac{255}{511} = .\overline{011111111}$	$\{1,2,4,8,16,32,64,128,256\}$	$\{510,509,507,503,495,479,447,383,255\}$
-1.999491	$(\star,-,+,+,+,+,+,+,-)$	$\frac{254}{511} = .\overline{011111110}$	$\{3,6,12,24,48,96,192,384,257\}$	$\{508,505,499,487,463,415,319,127,254\}$
-1.998587	$(\star,-,+,+,+,+,+,-,-)$	$\frac{253}{511} = .\overline{011111101}$	$\{5,10,20,40,80,160,320,129,258\}$	$\{506,501,491,471,431,351,191,382,253\}$
-1.997223	$(\star,-,+,+,+,+,+,-,+)$	$\frac{252}{511} = .\overline{011111100}$	$\{7,14,28,56,112,224,448,385,259\}$	$\{504,497,483,455,399,287,63,126,252\}$
-1.995419	$(\star,-,+,+,+,+,-,-,+)$	$\frac{251}{511} = .\overline{011111011}$	$\{9,18,36,72,144,288,65,130,260\}$	$\{502,493,475,439,367,223,446,381,251\}$
-1.993130	$(\star,-,+,+,+,+,-,-,-)$	$\frac{250}{511} = .\overline{011111010}$	$\{11,22,44,88,176,352,193,386,261\}$	$\{500,489,467,423,335,159,318,125,250\}$
-1.990376	$(\star,-,+,+,+,+,-,+,-)$	$\frac{249}{511} = .\overline{011111001}$	$\{13,26,52,104,208,416,321,131,262\}$	$\{498,485,459,407,303,95,190,380,249\}$
-1.987004	$(\star,-,+,+,+,+,-,+,+)$	$\frac{248}{511} = .\overline{011111000}$	$\{15,30,60,120,240,480,449,387,263\}$	$\{496,481,451,391,271,31,62,124,248\}$
-1.983810	$(\star,-,+,+,+,-,-,+,+)$	$\frac{247}{511} = .\overline{011110111}$	$\{17,34,68,136,272,33,66,132,264\}$	$\{494,477,443,375,239,478,445,379,247\}$
-1.979458	$(\star,-,+,+,+,-,-,+,-)$	$\frac{246}{511} = .\overline{011110110}$	$\{19,38,76,152,304,97,194,388,265\}$	$\{492,473,435,359,207,414,317,123,246\}$
-1.974781	$(\star,-,+,+,+,-,-,-,-)$	$\frac{245}{511} = .\overline{011110101}$	$\{21,42,84,168,336,161,322,133,266\}$	$\{490,469,427,343,175,350,189,378,245\}$
-1.969419	$(\star,-,+,+,+,-,-,-,+)$	$\frac{244}{511} = .\overline{011110100}$	$\{23,46,92,184,368,225,450,389,267\}$	$\{488,465,419,327,143,286,61,122,244\}$
-1.964024	$(\star,-,+,+,+,-,+,-,+)$	$\frac{243}{511} = .\overline{011110011}$	$\{25,50,100,200,400,289,67,134,268\}$	$\{486,461,411,311,111,222,444,377,243\}$
-1.957325	$(\star,-,+,+,+,-,+,-,-)$	$\frac{242}{511} = .\overline{011110010}$	$\{27,54,108,216,432,353,195,390,269\}$	$\{484,457,403,295,79,158,316,121,242\}$
-1.949575	$(\star,-,+,+,+,-,+,+,-)$	$\frac{241}{511} = .\overline{011110001}$	$\{29,58,116,232,464,417,323,135,270\}$	$\{482,453,395,279,47,94,188,376,241\}$
-1.932244	$(\star,-,+,+,-,-,+,+,-)$	$\frac{238}{511} = .\overline{011101110}$	$\{35,70,140,280,49,98,196,392,273\}$	$\{476,441,371,231,462,413,315,119,238\}$
-1.922286	$(\star,-,+,+,-,-,+,-,-)$	$\frac{237}{511} = .\overline{011101101}$	$\{37,74,148,296,81,162,324,137,274\}$	$\{474,437,363,215,430,349,187,374,237\}$
-1.911446	$(\star,-,+,+,-,-,+,-,+)$	$\frac{236}{511} = .\overline{011101100}$	$\{39,78,156,312,113,226,452,393,275\}$	$\{472,433,355,199,398,285,59,118,236\}$
-1.903117	$(\star,-,+,+,-,-,-,-,+)$	$\frac{235}{511} = .\overline{011101011}$	$\{41,82,164,328,145,290,69,138,276\}$	$\{470,429,347,183,366,221,442,373,235\}$
-1.890775	$(\star,-,+,+,-,-,-,-,-)$	$\frac{234}{511} = .\overline{011101010}$	$\{43,86,172,344,177,354,197,394,277\}$	$\{468,425,339,167,334,157,314,117,234\}$
-1.878383	$(\star,-,+,+,-,-,-,+,-)$	$\frac{233}{511} = .\overline{011101001}$	$\{45,90,180,360,209,418,325,139,278\}$	$\{466,421,331,151,302,93,186,372,233\}$
-1.841289	$(\star,-,+,+,-,+,-,+,-)$	$\frac{230}{511} = .\overline{011100110}$	$\{51,102,204,408,305,99,198,396,281\}$	$\{460,409,307,103,206,412,313,115,230\}$
-1.822756	$(\star,-,+,+,-,+,-,-,-)$	$\frac{229}{511} = .\overline{011100101}$	$\{53,106,212,424,337,163,326,141,282\}$	$\{458,405,299,87,174,348,185,370,229\}$
-1.785866	$(\star,-,+,+,-,+,-,-,+)$	$\frac{228}{511} = .\overline{011100100}$	$\{55,110,220,440,369,227,454,397,283\}$	$\{456,401,291,71,142,284,57,114,228\}$
-1.690142	$(\star,-,+,-,-,+,-,-,-)$	$\frac{218}{511} = .\overline{011011010}$	$\{75,150,300,89,178,356,201,402,293\}$	$\{436,361,211,422,333,155,310,109,218\}$
-1.656133	$(\star,-,+,-,-,+,-,+,-)$	$\frac{217}{511} = .\overline{011011001}$	$\{77,154,308,105,210,420,329,147,294\}$	$\{434,357,203,406,301,91,182,364,217\}$
-1.595681	$(\star,-,+,-,-,-,-,+,-)$	$\frac{214}{511} = .\overline{011010110}$	$\{83,166,332,153,306,101,202,404,297\}$	$\{428,345,179,358,205,410,309,107,214\}$
-1.555283	$(\star,-,+,-,-,-,-,-,-)$	$\frac{213}{511} = .\overline{011010101}$	$\{85,170,340,169,338,165,330,149,298\}$	$\{426,341,171,342,173,346,181,362,213\}$

A.10. **Period 10.**

$$\deg(G_{10}) = 495 \quad \text{and} \quad \gamma_{10} = 51.$$

−1.999986	$(\star,-,+,+,+,+,+,+,+,+)$	$\frac{511}{1023}=.\overline{0111111111}$	{1, 2, 4, 8, 16, 32, 64, 128, 256, 512} {1022, 1021, 1019, 1015, 1007, 991, 959, 895, 767, 511}
−1.999873	$(\star,-,+,+,+,+,+,+,+,-)$	$\frac{510}{1023}=.\overline{0111111110}$	{3, 6, 12, 24, 48, 96, 192, 384, 768, 513} {1020, 1017, 1011, 999, 975, 927, 831, 639, 255, 510}
−1.999647	$(\star,-,+,+,+,+,+,+,-,-)$	$\frac{509}{1023}=.\overline{0111111101}$	{5, 10, 20, 40, 80, 160, 320, 640, 257, 514} {1018, 1013, 1003, 983, 943, 863, 703, 383, 766, 509}
−1.999308	$(\star,-,+,+,+,+,+,+,-,+)$	$\frac{508}{1023}=.\overline{0111111100}$	{7, 14, 28, 56, 112, 224, 448, 896, 769, 515} {1016, 1009, 995, 967, 911, 799, 575, 127, 254, 508}
−1.998856	$(\star,-,+,+,+,+,+,-,-,+)$	$\frac{507}{1023}=.\overline{0111111011}$	{9, 18, 36, 72, 144, 288, 576, 129, 258, 516} {1014, 1005, 987, 951, 879, 735, 447, 894, 765, 507}
−1.998289	$(\star,-,+,+,+,+,+,-,-,-)$	$\frac{506}{1023}=.\overline{0111111010}$	{11, 22, 44, 88, 176, 352, 704, 385, 770, 517} {1012, 1001, 979, 935, 847, 671, 319, 638, 253, 506}
−1.997608	$(\star,-,+,+,+,+,+,-,+,-)$	$\frac{505}{1023}=.\overline{0111111001}$	{13, 26, 52, 104, 208, 416, 832, 641, 259, 518} {1010, 997, 971, 919, 815, 607, 191, 382, 764, 505}
−1.996805	$(\star,-,+,+,+,+,+,-,+,+)$	$\frac{504}{1023}=.\overline{0111111000}$	{15, 30, 60, 120, 240, 480, 960, 897, 771, 519} {1008, 993, 963, 903, 783, 543, 63, 126, 252, 504}
−1.995924	$(\star,-,+,+,+,+,-,-,+,+)$	$\frac{503}{1023}=.\overline{0111110111}$	{17, 34, 68, 136, 272, 544, 65, 130, 260, 520} {1006, 989, 955, 887, 751, 479, 958, 893, 763, 503}
−1.994889	$(\star,-,+,+,+,+,-,-,+,-)$	$\frac{502}{1023}=.\overline{0111110110}$	{19, 38, 76, 152, 304, 608, 193, 386, 772, 521} {1004, 985, 947, 871, 719, 415, 830, 637, 251, 502}
−1.993748	$(\star,-,+,+,+,+,-,-,-,-)$	$\frac{501}{1023}=.\overline{0111110101}$	{21, 42, 84, 168, 336, 672, 321, 642, 261, 522} {1002, 981, 939, 855, 687, 351, 702, 381, 762, 501}
−1.992479	$(\star,-,+,+,+,+,-,-,-,+)$	$\frac{500}{1023}=.\overline{0111110100}$	{23, 46, 92, 184, 368, 736, 449, 898, 773, 523} {1000, 977, 931, 839, 655, 287, 574, 125, 250, 500}
−1.991121	$(\star,-,+,+,+,+,-,+,-,+)$	$\frac{499}{1023}=.\overline{0111110011}$	{25, 50, 100, 200, 400, 800, 577, 131, 262, 524} {998, 973, 923, 823, 623, 223, 446, 892, 761, 499}
−1.989601	$(\star,-,+,+,+,+,-,+,-,-)$	$\frac{498}{1023}=.\overline{0111110010}$	{27, 54, 108, 216, 432, 864, 705, 387, 774, 525} {996, 969, 915, 807, 591, 159, 318, 636, 249, 498}
−1.987941	$(\star,-,+,+,+,+,-,+,+,-)$	$\frac{497}{1023}=.\overline{0111110001}$	{29, 58, 116, 232, 464, 928, 833, 643, 263, 526} {994, 965, 907, 791, 559, 95, 190, 380, 760, 497}
−1.985482	$(\star,-,+,+,+,+,-,+,+,+)$	$\frac{496}{1023}=\frac{16}{33}=.\overline{0111110000}$	{992, 961, 899, 775, 527, 31, 62, 124, 248, 496}
−1.982719	$(\star,-,+,+,+,-,-,+,+,-)$	$\frac{494}{1023}=.\overline{0111101110}$	{35, 70, 140, 280, 560, 97, 194, 388, 776, 529} {988, 953, 883, 743, 463, 926, 829, 635, 247, 494}
−1.980577	$(\star,-,+,+,+,-,-,+,-,-)$	$\frac{493}{1023}=.\overline{0111101101}$	{37, 74, 148, 296, 592, 161, 322, 644, 265, 530} {986, 949, 875, 727, 431, 862, 701, 379, 758, 493}
−1.978293	$(\star,-,+,+,+,-,-,+,-,+)$	$\frac{492}{1023}=.\overline{0111101100}$	{39, 78, 156, 312, 624, 225, 450, 900, 777, 531} {984, 945, 867, 711, 399, 798, 573, 123, 246, 492}
−1.976042	$(\star,-,+,+,+,-,-,-,-,+)$	$\frac{491}{1023}=.\overline{0111101011}$	{41, 82, 164, 328, 656, 289, 578, 133, 266, 532} {982, 941, 859, 695, 367, 734, 445, 890, 757, 491}
−1.973497	$(\star,-,+,+,+,-,-,-,-,-)$	$\frac{490}{1023}=.\overline{0111101010}$	{43, 86, 172, 344, 688, 353, 706, 389, 778, 533} {980, 937, 851, 679, 335, 670, 317, 634, 245, 490}
−1.970858	$(\star,-,+,+,+,-,-,-,+,-)$	$\frac{489}{1023}=.\overline{0111101001}$	{45, 90, 180, 360, 720, 417, 834, 645, 267, 534} {978, 933, 843, 663, 303, 606, 189, 378, 756, 489}
−1.967743	$(\star,-,+,+,+,-,-,-,+,+)$	$\frac{488}{1023}=.\overline{0111101000}$	{47, 94, 188, 376, 752, 481, 962, 901, 779, 535} {976, 929, 835, 647, 271, 542, 61, 122, 244, 488}
−1.965822	$(\star,-,+,+,+,-,+,-,+,+)$	$\frac{487}{1023}=.\overline{0111100111}$	{49, 98, 196, 392, 784, 545, 67, 134, 268, 536} {974, 925, 827, 631, 239, 478, 956, 889, 755, 487}
−1.962379	$(\star,-,+,+,+,-,+,-,+,-)$	$\frac{486}{1023}=.\overline{0111100110}$	{51, 102, 204, 408, 816, 609, 195, 390, 780, 537} {972, 921, 819, 615, 207, 414, 828, 633, 243, 486}
−1.959098	$(\star,-,+,+,+,-,+,-,-,-)$	$\frac{485}{1023}=.\overline{0111100101}$	{53, 106, 212, 424, 848, 673, 323, 646, 269, 538} {970, 917, 811, 599, 175, 350, 700, 377, 754, 485}
−1.955423	$(\star,-,+,+,+,-,+,-,-,+)$	$\frac{484}{1023}=.\overline{0111100100}$	{55, 110, 220, 440, 880, 737, 451, 902, 781, 539} {968, 913, 803, 583, 143, 286, 572, 121, 242, 484}
−1.951900	$(\star,-,+,+,+,-,+,+,-,+)$	$\frac{483}{1023}=.\overline{0111100011}$	{57, 114, 228, 456, 912, 801, 579, 135, 270, 540} {966, 909, 795, 567, 111, 222, 444, 888, 753, 483}
−1.946873	$(\star,-,+,+,+,-,+,+,-,-)$	$\frac{482}{1023}=.\overline{0111100010}$	{59, 118, 236, 472, 944, 865, 707, 391, 782, 541} {964, 905, 787, 551, 79, 158, 316, 632, 241, 482}
−1.935391	$(\star,-,+,+,-,-,+,+,-,-)$	$\frac{477}{1023}=.\overline{0111011101}$	{69, 138, 276, 552, 81, 162, 324, 648, 273, 546} {954, 885, 747, 471, 942, 861, 699, 375, 750, 477}
−1.929320	$(\star,-,+,+,-,-,+,+,-,+)$	$\frac{476}{1023}=.\overline{0111011100}$	{71, 142, 284, 568, 113, 226, 452, 904, 785, 547} {952, 881, 739, 455, 910, 797, 571, 119, 238, 476}
−1.925034	$(\star,-,+,+,-,-,+,-,-,+)$	$\frac{475}{1023}=.\overline{0111011011}$	{73, 146, 292, 584, 145, 290, 580, 137, 274, 548} {950, 877, 731, 439, 878, 733, 443, 886, 749, 475}
−1.919635	$(\star,-,+,+,-,-,+,-,-,-)$	$\frac{474}{1023}=.\overline{0111011010}$	{75, 150, 300, 600, 177, 354, 708, 393, 786, 549} {948, 873, 723, 423, 846, 669, 315, 630, 237, 474}
−1.914480	$(\star,-,+,+,-,-,+,-,+,-)$	$\frac{473}{1023}=.\overline{0111011001}$	{77, 154, 308, 616, 209, 418, 836, 649, 275, 550} {946, 869, 715, 407, 814, 605, 187, 374, 748, 473}
−1.899832	$(\star,-,+,+,-,-,-,-,+,-)$	$\frac{470}{1023}=.\overline{0111010110}$	{83, 166, 332, 664, 305, 610, 197, 394, 788, 553} {940, 857, 691, 359, 718, 413, 826, 629, 235, 470}
−1.894002	$(\star,-,+,+,-,-,-,-,-,-)$	$\frac{469}{1023}=.\overline{0111010101}$	{85, 170, 340, 680, 337, 674, 325, 650, 277, 554} {938, 853, 683, 343, 686, 349, 698, 373, 746, 469}
−1.887172	$(\star,-,+,+,-,-,-,-,-,+)$	$\frac{468}{1023}=.\overline{0111010100}$	{87, 174, 348, 696, 369, 738, 453, 906, 789, 555} {936, 849, 675, 327, 654, 285, 570, 117, 234, 468}
−1.882408	$(\star,-,+,+,-,-,-,+,-,+)$	$\frac{467}{1023}=.\overline{0111010011}$	{89, 178, 356, 712, 401, 802, 581, 139, 278, 556} {934, 845, 667, 311, 622, 221, 442, 884, 745, 467}
−1.874315	$(\star,-,+,+,-,-,-,+,-,-)$	$\frac{466}{1023}=.\overline{0111010010}$	{91, 182, 364, 728, 433, 866, 709, 395, 790, 557} {932, 841, 659, 295, 590, 157, 314, 628, 233, 466}
−1.861558	$(\star,-,+,+,-,-,-,+,+,-)$	$\frac{465}{1023}=\frac{5}{11}=.\overline{0111010001}$	{930, 837, 651, 279, 558, 93, 186, 372, 744, 465}
−1.846627	$(\star,-,+,+,-,+,-,+,-,-)$	$\frac{461}{1023}=.\overline{0111001101}$	{101, 202, 404, 808, 593, 163, 326, 652, 281, 562} {922, 821, 619, 215, 430, 860, 697, 371, 742, 461}
−1.835159	$(\star,-,+,+,-,+,-,+,-,+)$	$\frac{460}{1023}=.\overline{0111001100}$	{103, 206, 412, 824, 625, 227, 454, 908, 793, 563} {920, 817, 611, 199, 398, 796, 569, 115, 230, 460}
−1.829510	$(\star,-,+,+,-,+,-,-,-,+)$	$\frac{459}{1023}=.\overline{0111001011}$	{105, 210, 420, 840, 657, 291, 582, 141, 282, 564} {918, 813, 603, 183, 366, 732, 441, 882, 741, 459}
−1.816295	$(\star,-,+,+,-,+,-,-,-,-)$	$\frac{458}{1023}=.\overline{0111001010}$	{107, 214, 428, 856, 689, 355, 710, 397, 794, 565} {916, 809, 595, 167, 334, 668, 313, 626, 229, 458}
−1.802436	$(\star,-,+,+,-,+,-,-,+,-)$	$\frac{457}{1023}=.\overline{0111001001}$	{109, 218, 436, 872, 721, 419, 838, 653, 283, 566} {914, 805, 587, 151, 302, 604, 185, 370, 740, 457}
−1.721915	$(\star,-,+,-,-,+,-,-,+,-)$	$\frac{438}{1023}=.\overline{0110110110}$	{147, 294, 588, 153, 306, 612, 201, 402, 804, 585} {876, 729, 435, 870, 717, 411, 822, 621, 219, 438}
−1.701701	$(\star,-,+,-,-,+,-,-,-,-)$	$\frac{437}{1023}=.\overline{0110110101}$	{149, 298, 596, 169, 338, 676, 329, 658, 293, 586} {874, 725, 427, 854, 685, 347, 694, 365, 730, 437}
−1.629433	$(\star,-,+,-,-,+,-,+,-,-)$	$\frac{434}{1023}=\frac{14}{33}=.\overline{0110110010}$	{868, 713, 403, 806, 589, 155, 310, 620, 217, 434}
−1.536243	$(\star,-,+,-,-,-,-,-,-,-)$	$\frac{426}{1023}=.\overline{0110101010}$	{171, 342, 684, 345, 690, 357, 714, 405, 810, 597} {852, 681, 339, 678, 333, 666, 309, 618, 213, 426}
−1.501717	$(\star,-,+,-,-,-,-,-,+,-)$	$\frac{425}{1023}=.\overline{0110101001}$	{173, 346, 692, 361, 722, 421, 842, 661, 299, 598} {850, 677, 331, 662, 301, 602, 181, 362, 724, 425}
−1.447009	$(\star,-,+,-,-,-,+,-,+,-)$	$\frac{422}{1023}=.\overline{0110100110}$	{179, 358, 716, 409, 818, 613, 203, 406, 812, 601} {844, 665, 307, 614, 205, 410, 820, 617, 211, 422}

References

[B] T. Bousch, *Sur quelques problèmes de dynamique holomorphe*, Ph.D. thesis, Université de Paris-Sud, Orsay, 1992.

[BEK] X. Buff, A.L. Epstein & S. C. Koch, *Prefixed curves in moduli space*, preprint 2019, arXiv:1806.11221.

[G1] V. Goksel, *On the orbit of a post-critically finite polynomial of the form* $x^d + c$, Funct. Approx. Comment. Math., Volume 62, Number 1 (2020), pp. 95–104.

[G2] V. Goksel, *A note on Misiurewicz polynomials*, Journal de Théorie des Nombres de Bordeaux, Tome 32 (2020) no. 2, pp. 373–385.

[L1] M. Lutzky, *Counting stable cycles in unimodal iterations*, Physics Letters A, 131–4 (1988), pp. 248–250.

[L2] M. Lutzky, *Counting hyperbolic components of the Mandelbrot set*, Physics Letters A, 177–4 (1993), pp. 338–340.

[M] J. Milnor, *Arithmetic of unicritical polynomial maps*, Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday, Princeton Univ. Press (2012), pp. 15–23.

[MT] J. Milnor & W. Thurston, *On iterated maps of the interval*, Dynamical systems (College Park, MD, 1986–87), Lecture Notes in Mathematics, 1342, Berlin: Springer, pp. 465–563.

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