ON THE DECK GROUPS OF ITERATES OF BICRITICAL RATIONAL MAPS

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Abstract. A rational map on the Riemann sphere $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is said to be *bicritical* provided that f has exactly two critical points. In this note, we give a complete description of which groups (up to isomorphism) arise as the groups of deck transformations of iterates of bicritical rational maps. Our results generalize those from a previous paper by the authors.

1. Introduction

In this note, we study particular subgroups of the group Möb of Möbius transformations.

Definition 1.1. *Let f be a rational map. The deck group of f is*

$$\mathrm{Deck}(f) = \{\mu \in \textit{Möb} \mid f \circ \mu = f\},$$

made up of all $\mu \in$ Möb that preserve the fibers of f.

A partial analysis of the deck groups of iterates of bicritical rational maps was carried out in [4]. However, the goal of that work was to understand bicritical rational maps which shared an iterate, and so the deck groups of the iterates of such maps was only studied as a convenient tool. In the present work, we carry out a study of the deck groups of iterates of rational maps in earnest, and give a complete classification of which subgroups of the Möbius group can be realized as the deck group of an iterate of a bicritical rational map.

It is well-known that a finite group of Möbius transformations is isomorphic to either a cyclic group, a dihedral group, or one of the polyhedral groups A_4, A_5 or S_4 (see e.g [3]). In particular, if f is a rational map then for each k, the group $\mathrm{Deck}(f^k)$ is finite, and so must be one of the possibilities listed above. A study of rational maps of minimal degree with a given deck group[1] was carried out in [2]. In [6], Pakovich studies the groups $\mathrm{Deck}_\infty(f) = \bigcup_{k=1}^\infty \mathrm{Deck}(f^k)$ for rational maps f. He shows that, if f is not a power map, then $|\mathrm{Deck}_\infty(f)|$ is bounded, and this bound depends only on the degree d. He also computes some examples of $\mathrm{Deck}(f^k)$ for some given rational maps.

We will be concerned mainly with the case where the rational map f is bicritical. In this case, both the set $\mathcal{C}_f$ of critical points of f and the set

[1] In [2], elements of the deck group were called *half-symmetries*

of critical values $\mathcal{V}_f = f(\mathcal{C}_f)$ of f contain exactly two elements. In [4], the following result was proved.

Theorem 1.2. *Let f be a bicritical rational map and $k \in \mathbb{N}$. Then $\operatorname{Deck}(f^k)$ is either cyclic or dihedral. Furthermore, if the degree of f is odd, then $\operatorname{Deck}(f^k)$ is cyclic.*

We provide a sketch-proof of Theorem 1.2 in the present paper; the reader wishing to see the full details is referred to [4].

In the case where $\mathcal{C}_f = \mathcal{V}_f$, we will say that f is a power map. This is equivalent to f being conjugate to $z \mapsto z^{\pm d}$. In the case where f has degree 2, the following statement was obtained in [4].

Theorem 1.3. *If f is quadratic then the possibilities for $\operatorname{Deck}(f^k)$ (up to isomorphism) are $\mathbb{Z}_{2^n}$ for some $n \geq 1$, the Klein* Vierergruppe *V_4 or the dihedral group D_8 of order 8. Furthermore, if f is not a power map then $|\operatorname{Deck}(f^k)| \leq 8$.*

The main goal of the present note is to study the possibilities for $\operatorname{Deck}(f^k)$ where f is a bicritical rational map. We give a complete description of which groups (up to isomorphism) may be realized as $\operatorname{Deck}(f^k)$ for some bicritical rational map f.

Firstly, we will improve the result of Theorem 1.2 by showing that for odd degree bicritical maps, $\operatorname{Deck}(f^k) \cong \mathbb{Z}_d$, except if f is a power map, in which case $\operatorname{Deck}(f^k) \cong \mathbb{Z}_{d^k}$.

Theorem A. *Let f be a bicritical map of odd degree d. Then f is a power map if and only if there exists k such that $\operatorname{Deck}(f^k)$ is a cyclic group of order greater than d. In particular, if f is not a power map then $\operatorname{Deck}(f^k) \cong \mathbb{Z}_d$ for all $k \geq 1$.*

Our main theorem is the following, which generalizes Theorem 1.3 to even degree bicritical maps. Some of the methods generalize or make use of techniques first introduced in [4]. For completeness we include proofs in this article.

Theorem B. *Let f be a bicritical map of even degree d. The possibilities for $\operatorname{Deck}(f^k)$ (up to isomorphism) are $\mathbb{Z}_{d^n}$ for some $n \geq 1$, D_{2d} or D_{4d}. In particular, if f is not a power map then $|\operatorname{Deck}(f^k)| \leq 4d$.*

In Proposition 5.5, we give examples showing that the groups D_{2d} and D_{4d} may indeed be realized (up to isomorphism) as $\operatorname{Deck}(f^k)$ for some bicritical rational map f and some $k \geq 1$.

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2. Preliminaries on Möbius transformations and the Deck groups of rational maps

In this section we outline some standard results on Möbius transformations and deck groups of rational maps. This will allow us to analyze the deck groups of bicritical rational maps and outline the proof of Theorem 1.2. We hope that the statements in the present section may also in the future lend themselves to studying the deck groups of iterates of general rational maps.

2.1. **Deck groups and Möbius transformations.** A non-identity Möbius transformation $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of finite order has exactly two fixed points; we denote by $\mathrm{Fix}(\phi)$ the set of fixed point of ϕ. In [5], Milnor showed that a degree d bicritical map had order d rotational symmetry about its critical points, so that $\mathrm{Deck}(f) \cong \mathbb{Z}_d$. Furthermore, it is not hard to see that for any rational map, the group $\mathrm{Deck}(f^k)$ is a finite subgroup of Möb. Thus $\mathrm{Deck}(f^k)$ must be cyclic, dihedral, or isomorphic to one of the polyhedral groups A_4, A_5 or S_4. We collect these observations, and other standard facts about deck groups, into a Lemma. We denote by $\deg_z(f)$ the local degree of f at the point $z \in \hat{\mathbb{C}}$.

Lemma 2.1. *Let f be a rational map of degree $d \geq 1$.*

1. *The group $\mathrm{Deck}(f)$ is finite, and so must be cyclic, dihedral, or isomorphic to one of the polyhedral groups A_4, A_5 or S_4. Furthermore $|\mathrm{Deck}(f)| \leq d$.*
2. *Any non-identity element of $\mathrm{Deck}(f)$ has exactly two fixed points.*
3. *If $z \in \hat{\mathbb{C}}$ and $\phi \in \mathrm{Deck}(f)$, then $\deg_f(z) = \deg_f(\phi(z))$.*
4. *For all $k \geq 1$, $\mathrm{Deck}(f^k) \subseteq \mathrm{Deck}(f^{k+1})$.*
5. *If f is bicritical of degree d, then $\mathrm{Deck}(f)$ is cyclic of order d. Furthermore, each non-identity $\phi \in \mathrm{Deck}(f)$ has $\mathrm{Fix}(\phi) = \mathcal{C}_f$.*

We will often make use of the following classical result (see e.g. [1], Theorem 4.3.6) which characterizes exactly when two non-identity Möbius transformations commute.

Lemma 2.2. *Let ϕ and μ be non-identity Möbius transformations with fixed point sets $\mathrm{Fix}(\phi)$ and $\mathrm{Fix}(\mu)$ respectively. Then the following are equivalent.*

1. $\phi \circ \mu = \mu \circ \phi$
2. $\phi(\mathrm{Fix}(\mu)) = \mathrm{Fix}(\mu)$ *and* $\mu(\mathrm{Fix}(\phi)) = \mathrm{Fix}(\phi)$.
3. *Either* $\mathrm{Fix}(\mu) = \mathrm{Fix}(\phi)$ *or ϕ, μ and $\phi\circ\mu$ are involutions and* $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\mu) = \varnothing$.

2.2. **Sketch Proof of Theorem 1.2.** In this section, we provide an outline of the proof of Theorem 1.2 that was given in [4]. The following is the key observation.

Proposition 2.3. *Let f be a bicritical rational map of degree d, and let p be a prime number that does not divide d. Then for all natural numbers k, the group* $\mathrm{Deck}(f^k)$ *has no element of order p.*

(Sketch). If some element $\tau \in \mathrm{Deck}(f^k)$ were to have order p, then each element in $\hat{\mathbb{C}}$ would have orbit of length 1 or p under the action of $\langle \tau \rangle$. In particular the fiber $f^{-k}(w)$ over an regular value w contains d^k points, and since p does not divide d^k, the element τ must fix at least one element in such a fiber. But then as there are infinitely many regular points for f^k, we see that τ must be the identity, a contradiction. □

The proof of Theorem 1.2 now follows easily from Proposition 2.3.

Sketch proof of Theorem 1.2. The polyhedral groups A_4, A_5 and S_4 all contain elements of order 2 and elements of order 3. Thus if $\mathrm{Deck}(f^k)$ were polyhedral we would have$\deg(f) = d \geq 6$ by Proposition 2.3. But then by Lemma 2.1, then group $\mathrm{Deck}(f)$ would contain an element of order $d \geq 6$. But none of the polyhedral groups contain an element of order ≥ 6, so this is a contradiction.

Now suppose the degree of d is odd. In that case, 2 does not divide d and so by Proposition 2.3, $\mathrm{Deck}(f^k)$ cannot contain any element of order 2. Thus $\mathrm{Deck}(f^k)$ is not dihedral. □

2.3. **Deck groups and bicritical rational maps.** We now turn our attention to deck groups of iterates of bicritical rational maps. The following relatively simple observation has major ramifications which we will use in the rest of the paper. It generalizes an argument used in [4], and a similar result was given in [6].

Lemma 2.4. *Let f be a bicritical rational map with critical point set $\mathcal{C}_f$. Then if ϕ is a Möbius transformation such that $\phi(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique Möbius transformation μ such that $\mu \circ f = f \circ \phi$. Furthermore $\mu(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. Once we prove existence, the uniqueness will follow from the surjectivity of f. We first prove the existence result for $g(z) = z^d$. In this case, ϕ is a Möbius transformation such that $\phi(\mathcal{C}_g) = \mathcal{C}_g$ if and only if $\phi(z) = az^{\pm 1}$ for some $a \in \mathbb{C} \setminus \{0\}$. But then $g \circ \phi = a^d z^{\pm d}$, and so taking $\mu = a^d z$ completes the proof for $g(z) = z^d$.

Now suppose that f is bicritical of degree d. Then there exist Möbius transformations α and β such that $f = \alpha \circ g \circ \beta$, where $g(z) = z^d$. In particular $\beta(\mathcal{C}_g) = \mathcal{C}_f$. Thus if ϕ fixes $\mathcal{C}_f$ as a set then $\phi' = \beta^{-1} \circ \phi \circ \beta$ fixes $\mathcal{C}_g$ as a set, and by the above there exists μ' such that $\mu' \circ g = g \circ \phi'$. Hence taking $\mu = \alpha \circ \mu' \circ \alpha^{-1}$, a simple calculation yields $\mu \circ f = f \circ \phi$. The fact that $\mu(\mathcal{V}_F) = \mathcal{V}_F$ is clear. □

In particular, when the map ϕ in Lemma 2.4 belongs to $\mathrm{Deck}(f^k)$, we get the following.

Lemma 2.5. *Let f be a bicritical rational map and $\phi \in \mathrm{Deck}(f^k)$ for some k. If $\phi(\mathcal{C}_f) = \mathcal{C}_f$, then there exists a unique $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Moreover $\phi_{k-1}(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. Following Lemma 2.4, we only need to show that $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$. To see this, consider the following diagram.

$$\begin{array}{ccccc}
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} & \xrightarrow{f^{k-1}} & \hat{\mathbb{C}} \\
\big\downarrow{\scriptstyle \phi_k} & & \big\downarrow{\scriptstyle \phi_{k-1}} & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} & \xrightarrow{f^{k-1}} & \hat{\mathbb{C}}
\end{array}$$

The large outermost rectangle commutes since $\phi_k \in \mathrm{Deck}(f^k)$. Therefore, the square on the right commutes as well. As a consequence, $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$. □

In the next section, we will show that if f is a bicritical rational map, and $\phi \in \mathrm{Deck}(f^k)$, then $\phi(\mathcal{C}_f) = \mathcal{C}_f$. This will allow us to drop this hypothesis in Lemma 2.5 and strengthen it in Proposition 4.1. However, we will find the result of Lemma 2.5 useful when dealing with quadratic rational maps.

3. Elements of $\mathrm{Deck}(f^k)$ preserve the set of critical points of f

In this section we prove the following, which will be a key result in proving our two main theorems.

Theorem 3.1. *Let f be a bicritical rational map and $\phi \in \mathrm{Deck}(f^k)$ for some k. Then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

The proof in the case for $\deg(f) \geq 3$ is relatively simple. However, the quadratic case requires some more care.

3.1. **Quadratic rational maps.** The difficulty of proving Theorem 3.1 in the quadratic case comes from the complications arising when $\mathrm{Deck}(f^k)$ is dihedral. Accordingly, we start with some standard facts about dihedral groups. To fix our ideas, we use the following presentation of D_{2n}, the dihedral group of order $2n$.

$$D_{2n} = \langle R, F \mid R^n = F^2 = (RF)^2 = \mathrm{id} \rangle. \tag{1}$$

We include in this definition the case where $n = 2$, so that the Klein Vierergruppe V_4 is considered to be dihedral. We remark that if $\Gamma = \langle r \rangle$ is a cyclic subgroup of D_{2n} such that $|\Gamma| \geq 3$ then Γ is the unique such subgroup. Furthermore, there must exist ℓ such that $R^\ell = r$. The following simple group theoretic fact will be also useful.

Lemma 3.2. *Let n be even and suppose $\alpha \in D_{2n}$ has order 2. Then there exists a subgroup Γ of D_{2n} such that $\alpha \in \Gamma$ and $\Gamma \cong V_4$.*

Proof. The result is clearly true for the group V_4, so assume $n > 2$. Using the presentation (1), the center $Z(D_{2n})$ is isomorphic to $\mathbb{Z}_2$, and is generated by $\mu = R^{n/2}$. Thus if α is any other element of order 2 in D_{2n}, then $\Gamma = \{\mathrm{id}, \mu, \alpha, \mu\alpha\}$ forms a subgroup of D_{2n} isomorphic to V_4. □

Recall that by Lemma 2.1, then if f is bicritical of degree d then $\mathrm{Deck}(f) \cong \mathbb{Z}_d$. Furthermore, since $\mathrm{Deck}(f) \subseteq \mathrm{Deck}(f^k)$ for all k, all the $\mathrm{Deck}(f^k)$ contain a copy of $\mathbb{Z}_d$ as a subgroup. If $\mathrm{Deck}(f^k)$ is itself cyclic, then all non-identity elements of $\mathrm{Deck}(f^k)$ have the same pair of fixed points, which are necessarily the critical points of f. On the other hand, if $d \geq 3$, then by the discussion preceding Lemma 3.2, it follows that if $\mathrm{Deck}(f^k)$ is dihedral then $\mathrm{Deck}(f)$ is the unique cyclic subgroup of order d in $\mathrm{Deck}(f^k)$, and the generator of $\mathrm{Deck}(f)$ is an iterate of an element of maximal order in $\mathrm{Deck}(f^k)$. However, in the quadratic case we need to be more careful, since if $\mathrm{Deck}(f^k)$ is dihedral, it does not immediately follow that any order 2 subgroup of $\mathrm{Deck}(f^k)$ must be the group $\mathrm{Deck}(f)$.

Using terminology from [4], we say a bicritical rational map with critical values v_1 and v_2 is *critically coalescing* if $f(v_1) = f(v_2)$. Note that in this case, we must have $\mathcal{C}_f \cap \mathcal{V}_f = \varnothing$. For if v were both a critical point and a critical value of a degree d bicritical rational map which is critically coalescing, the image $f(v)$ would have at least $d+1$ preimages (counting multiplicity), which is impossible. Thus $f(v_1) = f(v_2)$ is equivalent to the condition that $f^k(c_1) = f^k(c_2)$ for all $k \geq 2$ (where c_1 and c_2 as usual denote the critical points of f). The next lemma is a slight generalization of a result from [4].

Lemma 3.3. *Let f be a bicritical rational map of even degree d. Then if $\mathrm{Deck}(f^k)$ is dihedral for some k, then f is critically coalescing.*

Proof. Write $\mathrm{Deck}(f) = \langle\tau\rangle$ and suppose $\mu = \tau^{d/2}$ is the unique element of order 2 in $\mathrm{Deck}(f)$. Suppose $k > 1$ is minimal such that $\mathrm{Deck}(f^k)$ is dihedral. By Lemma 3.2, there exists Γ, a subgroup of $\mathrm{Deck}(f^k)$ such that $\Gamma \cong V_4$ and $\mu \in \Gamma$. Write $\Gamma = \{\mathrm{id}, \mu, \alpha, \beta\}$. Since Γ is abelian and $\mathrm{Fix}(\mu) = \mathcal{C}_f$, then by Lemma 2.2 we have $\alpha(\mathcal{C}_f) = \beta(\mathcal{C}_f) = \mathcal{C}_f$. Thus, by Lemma 2.5, there exists $\nu \in \mathrm{Deck}(f^{k-1})$ such that $\nu \circ f = f \circ \alpha$. Furthermore, $\nu(\mathcal{V}_f) = \mathcal{V}_f = \{v_1, v_2\}$, and since α is an order 2 element distinct from μ, it is not an element of $\mathrm{Deck}(f)$. Thus we have $\nu \neq \mathrm{id}$. By the assumption on the minimality of k, $\mathrm{Deck}(f^{k-1})$ must be cyclic, and so for any non-identity elements $\gamma \in \mathrm{Deck}(f^{k-1})$ we have $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\mu) = \mathcal{C}_f$. Thus $\mathrm{Fix}(\nu) = \mathcal{C}_f$.

If ν fixes the elements of $\mathcal{V}_f$ pointwise, then we have $\mathcal{C}_f = \mathcal{V}_f$, and so f is a power map. But this is impossible, since $\mathrm{Deck}(f^k)$ is always cyclic for power maps. So ν must be an involution which exchanges the elements of $\mathcal{V}_f$. But since $\mathrm{Deck}(f^{k-1})$ is cyclic, it contains a unique involution, namely μ, and so $\nu = \mu$. Thus $\mu \in \mathrm{Deck}(f)$ interchanges the elements of $\mathcal{V}_f$, and so $f(v_1) = f(v_2)$. □

We remark that the converse of the above statement is true in degree 2 (see [4]). However, in higher degrees the converse does not hold.

Example 3.4. Let $f(z) = \frac{z^4-1}{z^4+i}$. Then $\mathcal{V}_f = \{1, i\}$ and $f(1) = f(i) = 0$, and so f is critically coalescing. One can readily check that $\mathrm{Deck}(f^2) = \mathrm{Deck}(f) \cong \mathbb{Z}_4$, and we will see later (Lemma 4.4) that this implies $\mathrm{Deck}(f^k) \cong \mathbb{Z}_4$ for all k.

Lemma 3.5. *Let F be a rational map and suppose* $\mathrm{Deck}(F)$ *contains an element of order k. Then there exists $z \in \hat{\mathbb{C}}$ such that* $\deg_z(F) \geq k$.

Proof. Let $\phi \in \mathrm{Deck}(F)$ have order k and let $z \in \mathrm{Fix}(\phi)$. Suppose that V is a simply connected neighborhood of $F(z)$ such that $V \cap \mathcal{V}_F \subseteq \{F(z)\}$, and let U be the component of $F^{-1}(V)$ which contains z. By restricting V if necessary, we may assume that $F^{-1}(F(z)) \cap U = \{z\}$. Then given $w \in V$, there exists $u_0 \in U$ such that $F(u_0) = w$. Since $F = F \circ \phi^j$ and $\phi^j(z) = z$ for all $0 \leq j \leq k-1$, we see that the component of $(F \circ \phi^j)^{-1}(V)$ which contains z is the set U. Thus for each j, $u_j = \phi^j(u_0)$ is in U. Hence F is (at least) k-to-1 on the set U, and the only critical point in U is z. Thus $\deg_z(F) \geq k$, by the Riemann-Hurwitz theorem. □

We are particularly interested in applying the previous result to the case where $F = f^k$ is an iterate of a quadratic rational map f and $\mathrm{Deck}(f^k)$ is dihedral.

Lemma 3.6. *Let f be a quadratic rational map and suppose* $\mathrm{Deck}(f^k)$ *is dihedral. If f^k has a critical point with local degree greater than 2, then one of the critical points c_1 of f is periodic of some period p. Furthermore:*

- *the second critical point c_2 satisfies $f^p(c_2) = c_1$ and*
- *for either critical point c, $f^n(c) = c_1$ if and only if $n = ap$ for some $a \geq 0$.*

Proof. By Lemma 3.3, f must be critically coalescing. Let the critical points of f be c_1 and c_2. Since f is critically coalescing, the image under f of a critical point of f cannot also be a critical point of f.

A point z maps forward with local degree greater than 1 under f^k if and only if z is a preimage $f^{-j}(c_i)$ for some $0 \leq j < k$ and $i = 1, 2$. Furthermore, if z maps forward by local degree strictly greater than 2 under f^k, then the forward orbit

$$\mathcal{O}_k(z) = (z, f(z), f^2(z), \ldots, f^{k-1}(z))$$

must contain (at least) two critical points of f. If the same critical point c_i appears twice, we are done, since then that critical point would be periodic. So assume without loss of generality that there exist $0 \leq n < m < k$ with $f^n(z) = c_2$ and $f^m(z) = c_1$. Then we have $f^{m-n}(c_2) = c_1$. But since f is critically coalescing, $f^\ell(c_1) = f^\ell(c_2)$ for all $\ell \geq 2$, whence $f^{m-n}(c_1) = c_1$ and so c_1 is a periodic critical point under f.

Now observe that if p is the period of c_1, so that $p > 0$ is minimal such that $f^p(c_1) = c_1$. Since f is critically coalescing, we also have $f^p(c_2) = c_1$, and if there were $0 < j < p$ such that $f^j(c_2) = c_1$, then this would imply $f^j(c_1) = c_1$, which is a contradiction. $\square$

Proposition 3.7. *Let f be a quadratic rational map and suppose that for some k the group* $\mathrm{Deck}(f^k)$ *is dihedral. Then if $\phi \in \mathrm{Deck}(f^k)$ then $\phi(\mathcal{C}_f) = \mathcal{C}_f$.*

Proof. If $\mathrm{Deck}(f^k)$ is isomorphic to V_4, then $\mathrm{Deck}(f^k)$ is abelian. Thus every element of $\mathrm{Deck}(f^k)$ commutes with μ, the unique order 2 element of $\mathrm{Deck}(f)$. Hence by Lemma 2.2, since $\mathrm{Fix}(\mu) = \mathcal{C}_f$ we have $\phi(\mathcal{C}_f) = \mathcal{C}_f$ for all $\phi \in \mathrm{Deck}(f^k)$.

We now assume that $\mathrm{Deck}(f^k)$ is dihedral and contains an element of order greater than 2. By Lemma 3.5, there must exist $z \in \hat{\mathbb{C}}$ such that $\deg_{f^k}(z) = k > 2$ and so by Lemma 3.6, f has a periodic critical point, c_1, and the other critical point c_2 eventually maps onto c_1, but is not in the forward orbit of c_1. We will show that the orbit $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$ under the action of $\mathrm{Deck}(f^k)$ is equal to $\mathcal{C}_f = \{c_1, c_2\}$. To see that $c_2 \in \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$, let Γ be a subgroup of $\mathrm{Deck}(f^k)$ which is isomorphic to V_4 and which contains μ (such a subgroup exists by Lemma 3.2). By Lemma 2.2, every non-identity element $\phi \in \Gamma$ must have $\phi(\mathcal{C}_f) = \mathcal{C}_f$. In particular, if $\phi \neq \mu$, then since μ and ϕ are distinct and both have order 2, we see that ϕ must transpose the elements of $\mathcal{C}_f$, and so $\phi(c_1) = c_2$, meaning $\mathcal{C}_f \subseteq \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$.

Now suppose $a \in \mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1)$, so that there exists $\phi \in \mathrm{Deck}(f^k)$ such that $\phi(c_1) = a$. We claim that a must eventually map onto c_1. Let $p \geq 2$ be the period of c_1. By Lemma 2.1, for all $j \geq 0$ and $1 \leq m \leq p$ such that $jp + m \geq k$ we have

$$\deg_{f^{jp+m}}(a) = \deg_{f^{jp+m}}(c_1) = 2^{j+1}. \tag{2}$$

Since $(\deg_{f^n}(a))_{n=1}^{\infty}$ is an unbounded increasing sequence, it follows that the forward orbit of a must contain infinitely many terms which are critical points. But since the only periodic critical point of f is c_1, we see that a must eventually map onto c_1.

We now show that $a \in \mathcal{C}_f$. Let j be minimal such that $jp + 1 \geq k$. Then by (2), we have $\deg_{f^{jp+m}}(a) = 2^{j+1}$ and so the orbit

$$\mathcal{O}_{jp}(a) = (a, f(a), f^2(a), \ldots, f^{jp}(a))$$

must contain $j+1$ critical points. Let the first appearance of a critical point be $f^q(a)$ for $q > 0$. Then by Lemma 3.6, there would be at most j critical points in the orbit $\mathcal{O}_{jp}(a)$. Thus a must itself be a critical point and so $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1) \subseteq \mathcal{C}_f$. We conclude that $\mathrm{orb}_{\mathrm{Deck}(f^k)}(c_1) = \mathcal{C}_f$. It follows that $\phi(\mathcal{C}_f) = \mathcal{C}_f$ for all $\phi \in \mathrm{Deck}(f^k)$. $\square$

Proof of Theorem 3.1. The result is trivial for $\phi = \mathrm{id}$, so suppose $\phi \neq \mathrm{id}$. If $\mathrm{Deck}(f^k)$ is cyclic, then every non-identity element of $\mathrm{Deck}(f^k)$ has the same set of fixed points, which must be equal to the set $\mathcal{C}_f$. Hence in this case $\phi(\mathcal{C}_f) = \mathcal{C}_f$.

On the other hand, suppose $\mathrm{Deck}(f^k)$ is dihedral. Then we know that the degree of f is even. If f is quadratic then the result holds by Proposition 3.7. Now suppose f has degree $d > 2$. If $\phi \in \mathrm{Deck}(f^k)$ is a power of some element of order greater than 2 then ϕ fixes the set $\mathcal{C}_f$ pointwise. The only other possibility is that ϕ has order 2. But then if $\rho \in \mathrm{Deck}(f^k)$ is an element of maximal order in $\mathrm{Deck}(f^k)$, by (1) we must have

$$\rho \circ \phi = \phi \circ \rho^{-1}. \tag{3}$$

Now suppose c is a critical point of f. Since $\rho^{-1}(c) = c$, applying (3) to c gives

$$\rho(\phi(c)) = \phi(\rho^{-1}(c)) = \phi(c)$$

from which it follows that $\phi(c)$ is a fixed point of ρ, and so $\phi(c)$ must be a critical point of f. Thus $\phi(c) \in \mathcal{C}_f$. □

4. Consequences of Theorem 3.1

Theorem 3.1 has a number of useful consequences. We first state a strengthened version of Lemma 2.5.

Proposition 4.1. *Let f be a bicritical rational map and $\phi_k \in \mathrm{Deck}(f^k)$ for some k. Then there exists a unique $\phi_{k-1} \in \mathrm{Deck}(f^{k-1})$ such that $f \circ \phi_k = \phi_{k-1} \circ f$. Moreover $\phi_{k-1}(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. The proof is the same as Lemma 2.5, with the hypothesis that $\phi_k(\mathcal{C}_f) = \mathcal{C}_f$ removed by Theorem 3.1. □

We now use Proposition 4.1 to prove a number of preliminary results which we will use to prove the main theorems. Observe that by Proposition 4.1, if for some $k > 1$ we have $\phi_k \in \mathrm{Deck}(f^k)$, then we can recursively define a sequence

$$(\phi_k, \phi_{k-1}, \ldots, \phi_1, \phi_0 = \mathrm{id})$$

where for each j, $\phi_j \in \mathrm{Deck}(f^j)$ and $f \circ \phi_j = \phi_{j-1} \circ f$. Each ϕ_j is uniquely determined by the initial choice of ϕ_k, and we must have $f^{k-j} \circ \phi_k = \phi_j \circ f^{k-j}$. This gives the following commutative diagram.

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_k"] & \hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_{k-1}"] & \ldots \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\phi_1"] & \hat{\mathbb{C}} \arrow[d, "\mathrm{id}"] \\
\hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] & \ldots \arrow[r, "f"] & \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}}
\end{tikzcd}$$

In particular, if $j = 1$ we obtain the following result.

Proposition 4.2. *Let f be a bicritical rational map. Let $k > 1$ and suppose $\phi_k \in \mathrm{Deck}(f^k)$. Then there exists a unique $\phi_1 \in \mathrm{Deck}(f)$ such that the following diagram commutes.*

$$\begin{array}{ccccc} \hat{\mathbb{C}} & \xrightarrow{f^{k-1}} & \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ {\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \mathrm{id}} \\ \hat{\mathbb{C}} & \xrightarrow{f^{k-1}} & \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \end{array}$$

Furthermore

(1) ϕ_1 *is the identity if and only if* $\phi_k \in \mathrm{Deck}(f^{k-1})$.
(2) $\phi_1(\mathcal{V}_f) = \mathcal{V}_f$.

Proof. By Proposition 4.1 and the discussion preceding the statement of the proposition, we know that there exists a unique $\phi_1 \in \mathrm{Deck}(f)$ such that $f^{k-1} \circ \phi_k = \phi_1 \circ f^{k-1}$. This proves the diagram commutes. Now suppose that $\phi_k \in \mathrm{Deck}(f^{k-1})$. Then we see that we must have $\phi_1 = \mathrm{id}$. On the other hand, if $\phi_1 = \mathrm{id}$, then the diagram shows that $f^{k-1} \circ \phi_k = f^{k-1}$, and so $\phi_k \in \mathrm{Deck}(f^{k-1})$. The assertion that $\phi_1(\mathcal{V}_f) = \mathcal{V}_f$ again follows from Proposition 4.1. □

It should be noted that in general, an element $\phi \in \mathrm{Deck}(f)$ need not map $\mathcal{V}_f$ to itself. For example, if $f(z) = \frac{1}{z^2-1}$, then the unique non-identity element of $\mathrm{Deck}(f)$ is $\phi(z) = -z$, which fixes the critical points 0 and ∞ of f. However, $\mathcal{V}_f = \{0, -1\}$, which is clearly not preserved by ϕ.

The following can be thought of as a partial converse to Proposition 4.1.

Lemma 4.3. *Let f be a bicritical rational map of degree d and suppose μ is a Möbius transformation such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. Then there exists a Möbius transformation ϕ such that $f \circ \phi = \mu \circ f$ and $\phi(\mathcal{C}_f) = \mathcal{C}_f$. In particular, if $\mu \in \mathrm{Deck}(f^k)$ for some $k \geq 1$ then $\phi \in \mathrm{Deck}(f^{k+1})$.*

Proof. The proof proceeds like that of Lemma 2.4, but note in this case there is no uniqueness. First, suppose $g(z) = z^d$. Then if $\mu(\mathcal{V}_g) = \mathcal{V}_g$, we have $\mu(z) = az^{\pm 1}$. Thus taking $\phi(z) = a^d z^{\pm 1}$, we get $g \circ \phi = \mu \circ g$ and $\phi(\mathcal{C}_g) = \mathcal{C}_g$ as required.

For the general case, we again note that if f is a bicritical rational map of degree d, then there exist Möbius transformations α and β such that $f = \alpha \circ g \circ \beta$ for $g(z) = z^d$. Thus if $\mu(\mathcal{V}_f) = \mathcal{V}_f$, then $\mu' = \alpha^{-1} \circ \mu \circ \alpha$ must satisfy $\mu'(\mathcal{V}_g) = \mathcal{V}_g$. Hence there exists ϕ' such that $g \circ \phi' = \mu' \circ g$ and $\phi'(\mathcal{C}_g) = \mathcal{C}_g$. Thus taking $\phi = \beta^{-1} \circ \phi' \circ \beta$ we get $f \circ \phi = \mu \circ f$ and $\phi(\mathcal{C}_f) = \mathcal{C}_f$, as required.

Finally, if $\mu \in \mathrm{Deck}(f^k)$ then since $f \circ \phi = \mu \circ f$, composing on the left by f^k gives

$$f^{k+1} \circ \phi = f^k \circ (f \circ \phi) = f^k \circ (\mu \circ f) = (f^k \circ \mu) \circ f = f^k \circ f = f^{k+1}$$

and so $\phi \in \mathrm{Deck}(f^{k+1})$. □

Lemma 4.4. *Suppose f is a rational map. Suppose for some k that* $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^{k+1})$. *Then* $\mathrm{Deck}(f^{k+2}) = \mathrm{Deck}(f^{k+1}) = \mathrm{Deck}(f^k)$.

Proof. Let $\phi \in \operatorname{Deck}(f^{k+2})$. By Proposition 4.1, there exists $\mu \in \operatorname{Deck}(f^{k+1})$ such that

$$f \circ \phi = \mu \circ f. \tag{4}$$

Since $\operatorname{Deck}(f^{k+1}) = \operatorname{Deck}(f^k)$, we see that $\mu \in \operatorname{Deck}(f^k)$. But then post-composing (4) by f^k yields

$$f^{k+1} \circ \phi = f^k \circ (\mu \circ f) = (f^k \circ \mu) \circ f = f^{k+1}$$

and so $\phi \in \operatorname{Deck}(f^{k+1})$. □

Lemma 4.4 ensures that the function of Example 3.4 has $\operatorname{Deck}(f^k) \cong \mathbb{Z}_4$ for all $k \geq 1$.

We will use the notation $\operatorname{Deck}^*(f^k) = \operatorname{Deck}(f^k) \setminus \operatorname{Deck}(f^{k-1})$, with the convention that $\operatorname{Deck}(f^0) = \operatorname{Deck}^*(f^0) = \{\mathrm{id}\}$.

Lemma 4.5. *Let f be a bicritical rational map and $k \geq 0$. Then $\operatorname{Deck}^*(f^{k+1}) \neq \varnothing$ if and only if there exists $\mu \in \operatorname{Deck}^*(f^k)$ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$.*

Proof. Suppose $\phi \in \operatorname{Deck}^*(f^{k+1})$. By Proposition 4.1, there exists $\mu \in \operatorname{Deck}(f^k)$ such that

$$\mu \circ f = f \circ \phi \tag{5}$$

and $\mu(\mathcal{V}_f) = \mathcal{V}_f$. If μ is not an element of $\operatorname{Deck}^*(f^k)$, then $\mu \in \operatorname{Deck}(f^{k-1})$. Thus $f^{k-1} \circ \mu = f^{k-1}$, and so composing f^{k-1} on the left of (5) we get

$$f^k = f^{k-1} \circ \mu \circ f = f^k \circ \phi$$

and so $\phi \in \operatorname{Deck}(f^k)$. But this contradicts $\phi \in \operatorname{Deck}^*(f^{k+1})$, and so we conclude that $\mu \in \operatorname{Deck}^*(f^k)$.

Conversely, suppose that there exists $\mu \in \operatorname{Deck}^*(f^k)$ such that $\mu(\mathcal{V}_f) = \mathcal{V}_f$. It follows from Lemma 4.3 that there exists $\phi \in \operatorname{Deck}(f^{k+1})$ such that $\mu \circ f = f \circ \phi$. Suppose that $\phi \in \operatorname{Deck}(f^k)$. Then Proposition 4.1 asserts that there is a unique $\mu' \in \operatorname{Deck}(f^{k-1})$ such that $\mu' \circ f = f \circ \phi$. But then $\mu = \mu'$, and so this contradicts $\mu \in \operatorname{Deck}^*(f^k)$. Hence $\phi \in \operatorname{Deck}^*(f^{k+1})$ and so $\operatorname{Deck}^*(f^{k+1}) \neq \varnothing$. □

Lemma 4.6. *Let f be a bicritical rational map. Then*

$$\frac{|\operatorname{Deck}(f^k)|}{|\operatorname{Deck}(f^{k-1})|} \leq d. \tag{6}$$

Furthermore, if f is not a power map, then the quotient is at most 2.

Proof. Suppose $\phi \in \operatorname{Deck}^*(f^k)$. Then by Proposition 4.2, there exists a unique non-identity element $\mu \in \operatorname{Deck}(f)$ such that $f^{k-1} \circ \phi = \mu \circ f^{k-1}$.

Define $h \colon \operatorname{Deck}(f^k) \to \operatorname{Deck}(f)$ by $h(\phi) = \mu$, where μ is defined as the map from the above paragraph. We claim that h is a homomorphism. To see this, note that if $f^{k-1} \circ \phi_1 = \mu_1 \circ f^{k-1}$ and $f^{k-1} \circ \phi_2 = \mu_2 \circ f^{k-1}$, then

$$\begin{aligned} f^{k-1} \circ \phi_1 \circ \phi_2 &= \mu_1 \circ f^{k-1} \circ \phi_2 \\ &= \mu_1 \circ \mu_2 \circ f^{k-1} \\ &= \mu_1 \circ \mu_2 \circ f^{k-1} \end{aligned}$$

It follows that $h(\phi_1 \circ \phi_2) = h(\phi_1) \circ h(\phi_2)$. Again by Proposition 4.2, we have $\ker h = \mathrm{Deck}(f^{k-1})$. Since each coset of $\ker h$ in $\mathrm{Deck}(f^k)$ has cardinality equal to $|\mathrm{Deck}(f^{k-1})|$, and since there are at most $|\mathrm{Deck}(f)| = d$ cosets, we conclude the first statement is true.

Now suppose f is not a power map. Then there is at most one non-identity element μ_1 of $\mathrm{Deck}(f)$ such that $\mu_1(\mathcal{V}_f) = \mathcal{V}_f$. Thus the image of h is a subgroup of $\{\mathrm{id}, \mu_1\}$, and so the quotient (6) is at most 2. $\square$

5. Proof of the Main Theorems

5.1. **Möbius transformations preserving the sets of critical points and critical values values of a bicritical rational map.** As can be ascertained from Proposition 4.1 and Lemma 4.5, the Möbius transformations μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ are of particular importance when it comes to analyzing the groups $\mathrm{Deck}(f^k)$. In fact, when f is not a power map these two conditions on μ are very restrictive.

Lemma 5.1. *Let f be a bicritical rational map of degree d such that $|\mathcal{C}_f \cup \mathcal{V}_f| = 3$. Then the only Möbius transformation μ satisfying $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ is the identity. Furthermore,* $\mathrm{Deck}(f^k) \cong \mathbb{Z}_d$ *for all $k \geq 1$*

Proof. Since $|\mathcal{C}_f \cap \mathcal{V}_f| = 3$, there exists a unique $w \in \mathcal{C}_f \cap \mathcal{V}_f$. But then any Möbius transformation μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ must fix w. Therefore, μ would have to act as the identity on the three element set $\mathcal{C}_f \cap \mathcal{V}_f$, and so $\mu = \mathrm{id}$. The final claim then follows from Lemmas 4.4 and 4.5. $\square$

Lemma 5.2. *Let f be a bicritical rational map such that $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$. Then there exist at most four Möbius transformations μ such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$. Any such μ which is not the identity must be an involution. Furthermore, these maps form a group under composition, and if they all exist, this group is isomorphic to V_4.*

Proof. Since a Möbius transformation is uniquely characterised by its action on three points, we see there are the following possibilities for μ.

(1) μ fixes the elements of $\mathcal{C}_f$ and $\mathcal{V}_f$ pointwise, so that $\mu = \mathrm{id}$.
(2) μ_1 such that $\mathrm{Fix}(\mu_1) = \mathcal{C}_f$ and μ_1 swaps the elements of $\mathcal{V}_f$.
(3) μ_2 such that $\mathrm{Fix}(\mu_2) = \mathcal{V}_f$ and μ_2 swaps the elements of $\mathcal{C}_f$.
(4) μ_3 such that μ_3 swaps the elements of $\mathcal{C}_f$ and the elements of $\mathcal{V}_f$.

Furthermore, if μ is not the identity, then μ is an involution. To complete the proof, we need to show that these four Möbius transformations form a group. But by Lemma 2.2, μ_1 and μ_2 commute. Furthermore, $\mu_1 \circ \mu_2 = \mu_3$, from which it follows that $\langle \mu_1, \mu_2 \rangle \cong V_4$. □

We will continue to use the notation μ_i, $i = 1, 2, 3$ to denote the transformations obtained from the above lemma. Since any element of $\mathrm{Deck}(f)$ fixes $\mathcal{C}_f$ pointwise, we have $\mu_2, \mu_3 \notin \mathrm{Deck}(f)$.

Lemma 5.3. *Let f be a bicritical rational map of degree d, and suppose f is not a power map. If* $\mathrm{Deck}^*(f^2) \neq \varnothing$ *then* $\mathrm{Deck}(f^2) \cong D_{2d}$.

Proof. By Lemma 4.6 we have $|\mathrm{Deck}(f^2)| = 2|\mathrm{Deck}(f)| = 2d$, and so by Theorem 1.2 we must have $\mathrm{Deck}(f^2) \cong \mathbb{Z}_{2d}$ or $\mathrm{Deck}(f^2) \cong D_{2d}$. Since $\mathrm{Deck}(f^2) \neq \mathrm{Deck}(f)$, we know from Lemma 4.5 that there exists a non-identity $\mu_1 \in \mathrm{Deck}(f)$ such that $\mu_1(\mathcal{V}_f) = \mathcal{V}_f$. Since μ_1 must fix the elements of $\mathcal{C}_f$ pointwise, we see that μ must swap the elements of $\mathcal{V}_f = \{v_1, v_2\}$. Thus $f(v_1) = f(v_2)$ and f is critically coalescing.

Now suppose that $\mathrm{Deck}(f^2) \cong \mathbb{Z}_{2d}$. By Lemma 3.5, there exists c such that $\deg_c(f^2) \geq 2d$. But then $\deg_c(f^2) = d^2$ and so $c \in \mathcal{C}_f \cap \mathcal{V}_f$. Therefore by Lemma 5.1, we have $\mathrm{Deck}(f^2) \cong \mathbb{Z}_d$, which is a contradiction. □

Proposition 5.4. *Let f be a bicritical rational map which is not a power map. Then* $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^3)$ *for all $k \geq 3$.*

Proof. Using the notation of Lemma 5.2, denote by $\Gamma = \mathrm{id}, \mu_1, \mu_2, \mu_3\}$ the group of Möbius transformations such that $\mu(\mathcal{C}_f) = \mathcal{C}_f$ and $\mu(\mathcal{V}_f) = \mathcal{V}_f$ (note that some of the μ_i may not exist, but those that do must form a group). By Lemma 4.4, we may assume that $\mathrm{Deck}(f) \subsetneq \mathrm{Deck}(f^2) \subsetneq \mathrm{Deck}(f^3)$, since otherwise the statement of the proposition holds.

Since $\mathrm{Deck}(f) \subsetneq \mathrm{Deck}(f^2)$, it follows from Lemma 4.5 that $\mu_1 \in \mathrm{Deck}^*(f)$. Similarly, since $\mathrm{Deck}(f^3) \neq \mathrm{Deck}(f^2)$, one of μ_2 or μ_3 is an element of $\mathrm{Deck}^*(f^2)$. But since $\mu_3 = \mu_1 \circ \mu_2$, we see that both μ_2 and μ_3 must belong to the group $\mathrm{Deck}(f^2)$. However, this means that $\Gamma \subseteq \mathrm{Deck}(f^2)$ and so $\mathrm{Deck}^*(f^3) \cap \Gamma = \varnothing$. But by Lemma 4.5 this means that $\mathrm{Deck}(f^4) = \mathrm{Deck}(f^3)$, and so by Lemma 4.4 we have $\mathrm{Deck}(f^k) = \mathrm{Deck}(f^3)$ for all $k \geq 3$. □

5.2. **Proof of Theorem A.**

Proof of Theorem A. It is clear that if f is a power map then $\mathrm{Deck}(f^k) \cong \mathbb{Z}_{d^k}$. Now suppose f is not a power map, so that $|\mathcal{C}_f \cup \mathcal{V}_f| > 2$. If $|\mathcal{C}_f \cup \mathcal{V}_f| = 3$, then Lemma 5.1 asserts that $\mathrm{Deck}(f^k) \cong \mathbb{Z}_d$ for all k. If $|\mathcal{C}_f \cup \mathcal{V}_f| = 4$ then we note that since d is odd, $\mathrm{Deck}(f) \cong \mathbb{Z}_d$ cannot contain an element of order 2. But this means none of the elements μ_i, $i = 1, 2, 3$ from Lemma 5.2 can belong to $\mathrm{Deck}(f)$, and so by Lemma 4.5 we have $\mathrm{Deck}(f^2) = \mathrm{Deck}(f) \cong \mathbb{Z}_d$. Thus by Lemma 4.4, we have $\mathrm{Deck}(f^k) = \mathrm{Deck}(f) \cong \mathbb{Z}_d$ for all $k \geq 1$. □

5.3. **Proof of Theorem B.**

Proof of Theorem B. It is clear that if f is a power map, then $\operatorname{Deck}(f^k) \cong \mathbb{Z}_{d^k}$ for all k. If f is not a power map, then by Lemmas 4.4 and 5.3, then either $\operatorname{Deck}(f^2) \cong D_{2d}$ or $\operatorname{Deck}(f^k) \cong \mathbb{Z}_d$ for all $k \geq 1$.

If $\operatorname{Deck}(f^2) \cong D_{2d}$, then by Lemma 4.6, the only options for $\operatorname{Deck}(f^3)$ up to isomorphism, are D_{4d} or D_{2d}. But by Proposition 5.4, the group $\operatorname{Deck}(f^k)$ cannot be larger than $\operatorname{Deck}(f^3)$, and this completes the proof. □

We conclude by showing that D_{2d} and D_{4d} are realized as $\operatorname{Deck}(f^k)$ for some rational map f.

Proposition 5.5. *Let d be even.*

(1) *If $f(z) = \frac{z^d - a}{z^d + a}$ for some $a \neq 0$ then* $\operatorname{Deck}(f^2) \cong D_{2d}$.
(2) *If $g(z) = \frac{z^d - 1}{z^d + 1}$ then* $\operatorname{Deck}(g^3) \cong D_{4d}$.

Proof. We note that since $\mathcal{C}_f = \mathcal{C}_g = \{0, \infty\}$, the involution $\mu(z) = -z$ belongs to $\operatorname{Deck}(f)$ and $\operatorname{Deck}(g)$.

(1) It is clear that $\mathcal{V}_f = \{-1, 1\}$, and since $\mu(\mathcal{V}_f) = \mathcal{V}_f$, it follows from Lemmas 4.5 and 5.3 that $\operatorname{Deck}(f^2) \cong D_{2d}$.
(2) Let $\phi(z) = \frac{1}{z}$. Then a simple calculation yields

$$g \circ \phi(z) = -\frac{z^d - 1}{z^d + 1} = \mu \circ g(z).$$

Since from the first part we know $\mu \in \operatorname{Deck}(f)$, we see that by Lemma 4.3 we must have $\phi \in \operatorname{Deck}(g^2)$. Furthermore, it is clear that $\phi(\mathcal{V}_g) = \mathcal{V}_g$ and so $\operatorname{Deck}^*(g^3) \neq \varnothing$ by Lemma 4.5. But then $|\operatorname{Deck}(g^3)| = 2|\operatorname{Deck}(g^2)|$ by Lemma 4.6 and so $\operatorname{Deck}(g^3) \cong D_{4d}$. □

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